NOISE IN COULOMB BLOCKADED QUANTUM DOTS AND KONDO SYSTEMS

by

DANIEL MIKAEL LARSSON

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Abstract

In this thesis we study the noise in Coulomb blockaded quantum dots in the vicinity of the peak of conductance in a full quantum treatment using the Keldysh technique. In previous work on this system, the emphasis have been on master equation approaches in the shot noise regime. In the vicinity of the peak of conductance it remained unclear if this classical approach is valid since we have two strongly interacting charging states and a full quantum treatment is necessary. Using our full, quantum mechanical approach we find an analytical expression for the noise valid from the low bias regime all the way to the shot noise regime valid in the vicinity of peak of conductance. In the shot noise regime we recover the result from the master equation approach.

Acknowledgements

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Contents

List of Figures

Chapter 1

INTRODUCTION

1.1 Structure of the thesis

This thesis is focused on noise in mesoscopic systems. In chapter 1 we give a non-technical introduction to the basic concepts covered in this thesis. The basic knowledge needed about path integrals and Green functions is covered in chapter 2. The first new results are introduced in chapter 3, where the tunneling density of states is derived in an approach introduced by Sedlmayr et al. [1], and then re-derived within a new approach developed for making the generalisation to noise possible. Chapter 4 is the core of this thesis, where the noise of a Coulomb blockaded quantum dot in calculated using the method developed in chapter 3. Constructive bosonisation is introduced in chapter 5 and will be used in chapter 6 where we discuss noise in Kondo system and suggest a new interesting problem for the future. We will in this thesis set $\hbar = k_B = 1$. We have also used the standard notation that unless the limits of the integrals are given they are from -∞ to ∞.

1.2 Mesoscopic systems

In this thesis we will study quantum transport in mesoscopic systems. These are systems that are sufficiently big for statistical physics to hold, but small enough to contain significant fluctuations. These length scales are today accessible thanks to the great development

in nano-fabrication technologies [2, 3, 4, 5]. In these structures the most important type of experiments are transport measurements. These experiments are an average of many readings from a measuring device. In these experiments each individual reading is random. This randomness has several sources, for example imperfections in the measuring device, temperature fluctuations or fluctuations that are due to quantum effects. The last type of fluctuations are the most interesting ones and the source of a lot of information about the correlations in electron systems.

1.3 Counting electrons

At this stage there are two approaches: either we average out all fluctuations or we study the whole statistics. The first alternative is the most common since it is the simpler to perform experimentally. For transport it gives the average current in the system. The second approach is to measure the statistics of the transported charge. This requires a lot more data, but has the benefit that it provides a lot more information about the system than the average current.

1.3.1 Some probability theory

To build a theory of the statistics of the fluctuations in the electron transfer in nanostructures [6] we start with reminding the reader about some basic concepts from probability theory. If we want to perform a measurement counting a random event during a time interval Δt, the outcome of the measurement, N, is a random number. If we perform several experiments we can obtain the average, $\langle N \rangle$, by adding all the outcomes and dividing by the number of measurements. Even more useful than the average is the probability distribution of the outcomes. The distribution gives the probability P_N to obtain an outcome N if a

measurement is performed on the system. The distribution is obtained by performing a number of experiments M_{tot} on a system. The probability to obtain outcome N is then the number of experiments, M_N, with the outcome N divided by the total number of experiments M_{tot}, I.e. $P_N = M_N/M_{tot}$.

$$\sum_N P_N = 1. \tag{1.3.1}$$

Using the distribution we can now calculate the average

$$\langle N \rangle = \sum_N N P_N, \tag{1.3.2}$$

and the variance of the distribution also know as the second cumulant is given as

$$\langle\langle N^2 \rangle\rangle = \langle (N - \langle N \rangle)^2 \rangle = \sum_N N^2 P_N - \left(\sum_N N P_N \right)^2. \tag{1.3.3}$$

In quantum transport theory, where N refers to the number of electrons that passes through the system in a given time interval Δt, the variance is usually referred to as the noise. In most situations this description of the distribution is not the most convenient. Let us introduce the characteristic function

$$\Lambda(\chi) = \left\langle e^{i\chi N} \right\rangle = \sum_N P_N e^{i\chi N}, \tag{1.3.4}$$

where we have introduced the measuring field χ. We can now obtain all the cumulants by taking derivatives of $\ln \Lambda(\chi)$ with respect to χ at the point $\chi = 0$. In appendix B we discuss what information can be gained from the different cumulants for a non-interacting electron systems. The characteristic function also has the convenient property that if two processes

are statistically independent, i.e.

$$P_N^{tot} = \sum_{M=0}^{N} P_M^2 P_{N-M}^1, \tag{1.3.5}$$

where P_M^i denotes the probability to obtain the value M in a measurement of process, $i = 1, 2$. The total characterisitic function is given by the product of the two characteristic functions, $\Lambda^{tot}(\chi) = \Lambda_1(\chi)\Lambda_2(\chi)$.

1.4 Noise

In a nano-structure we are interested in counting the number of electrons that transfer from one reservoir to another during a time interval Δt. The quantity we count is the charge Q and on average $Q = \Delta t I$ where I is the average current. We start with a simple system where electrons can only be transferred from the left to the right. The probability for one electron to be transferred during a small time interval, dt, is $\Gamma dt \ll 1$, and the probability that it will be reflected is $1 - \Gamma dt$, where we have introduced the transfer rate Γ. In this simple example we chose to neglect processes where more than one electron is transferred at the same time. The characteristic function is now given by

$$\Lambda_{dt}(\chi) = \left\langle e^{i\chi Q/e} \right\rangle = (1 - \Gamma dt) + \Gamma dt e^{i\chi}. \tag{1.4.1}$$

Since the electrons, with charge e, pass through independently, we can rewrite the characteristic function as a products of the individual events,

$$\Lambda_{\Delta t}(\chi) = (\Lambda_{dt}(\chi))^{\Delta t/dt} = \exp\left(\Gamma \Delta t \left(e^{i\chi} - 1\right)\right) = \exp\left(\tilde{N}\left(e^{i\chi} - 1\right)\right), \tag{1.4.2}$$

where $\tilde{N} = \Gamma \Delta t$ is the average number of electrons transferred. If we go back to the probabilities P_N by taking the inverse Fourier transform on the characteristic function we

obtain

$$\begin{aligned} P_N &= \int_0^{2\pi} \frac{d\chi}{2\pi} \Lambda(\chi) e^{-iN\chi} \approx \int_0^{2\pi} \frac{d\chi}{2\pi} e^{-i\chi N + \tilde{N}(e^{i\chi}-1)} \\ &= \frac{\tilde{N}^N}{N!} e^{-i\tilde{N}\Delta t \theta(N)}, \end{aligned} \tag{1.4.3}$$

which is the Poisson distribution. Here $\theta(N)$ is the unit step function. This idealised system does is fact exist in the form of tunneling junctions where the intervals between successive tunneling events are so long that the electrons are effectively non-interacting. The opposite limit is where we have perfect transmission T=1. In this situation the noise is zero. In this thesis we are interested in the intermediate regime where the tunneling is small but interactions play a large role. We start by defining the noise power spectrum [7, 8, 9] in terms of the current operator, I,

$$S(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} \langle \{\delta I(t), \delta I(0)\} \rangle. \tag{1.4.4}$$

Where $\{.,.\}$ denotes the anti commutator. The noise power spectrum consists of two parts one coming from thermal fluctuations called Nyquist-Johson noise [10, 11]. The other is the shot noise that originates the discreteness of the charge of the electrons [12]. Thermal noise in equilibrium is related to the conductance G by the fluctuation-dissipation relation

$$S = 4k_B T G, \tag{1.4.5}$$

as long as $\hbar\omega \ll k_B T$. From this it is clear that beyond the conductance we cannot obtain any new information about the transport from the thermal noise.

In contrast to the thermal noise, shot noise in a system driven out of equilibrium by an applied bias voltage V contains useful information about the correlations of the electrons which is not included in the conductance. In non-interacting fermionic systems we obtain

the maximal shot noise value

$$S_{Poisson} = 2eI, \tag{1.4.6}$$

which is proportional to the average current I. This Poisson limit is used as a reference point in interacting systems, and the Fano factor is defined as

$$f = \frac{S(0)}{2eI}. \tag{1.4.7}$$

In most systems the interactions suppress the noise, and we obtain a Fano factor less then 1. For instance, macroscopic conductors have zero shot noise because inelastic electron-phonon scattering averages out the current fluctuations. It is also common that the deviation from Poisson statistics is characterised by an effective charge

$$e^* = \frac{S(0)}{2I}. \tag{1.4.8}$$

This is discussed in more detail in chapter 6, where Kondo systems are studied. For non-interacting systems the Pauli principle, which forbids multiple occupancy of the same single-particle state, will introduce correlations. A typical example is a ballistic point contact where $S = 0$ because the stream of incoming electrons is completely correlated by the Pauli principle.

Over a very wide range, both the thermal and shot noise power do not depend on frequency, also known as a white noise spectrum. At low frequencies there is a third source of noise, the $1/f$ noise (or flicker noise) caused by random motion of impurities, which produce time-dependent fluctuations in the conductance. For low frequencies this noise is completely dominant and thus it is important to take into account when experiments are performed. But of course other types of noise will of also be present in experiments, and should be taken

into account.

1.5 Quantum dots

A quantum dot [4, 5, 6, 13, 14] is a small conducting device where up to several thousands electrons are confined in a region of linear size of about $0.1-1\mu m$. It is constructed by forming a two-dimensional electron gas in the interface region of a semiconductor heterostructure and applying metallic gates to further confine the electron number and size of the dot, see Fig. 1.1. Since the electron motion is confined in all spatial directions we will consider the dot zero-dimensional in this thesis. To measure the transport properties of the dot it will be coupled to leads, and a current will be driven through the system by an applied bias voltage V. The coupling to the leads can be experimentally controlled. There are two interesting situations, open and closed dots. In an open dot the coupling to the leads is very strong and the transport of electrons through the dot is classically allowed. In the closed or isolated quantum dot the point contacts are pinched off and effective barriers are formed, so that transport is only possible through tunneling. The number of electrons on the dot is described classically by the capacitance, C, of the dot. The energy required to add one electron is given by e^2/C. In a typical quantum dot made of GaAs, the charging energy for adding one additional electron $E_c = e^2/C \approx 1meV$ [13].

In this thesis we will focus on closed quantum dots, where the barriers are big enough for the transmission to be small, i.e. $G \ll e^2/\hbar$. The second condition that $kT \ll E_c \approx 1meV = 12K$ is to guarantee that the charging energy is not washed away. This condition is always fulfilled in the low temperatures used in experiments.

The quantisation of the energy levels in the closed quantum dot is one of the key differences between classical and quantum mechanics. We will take a look at the statistics of the

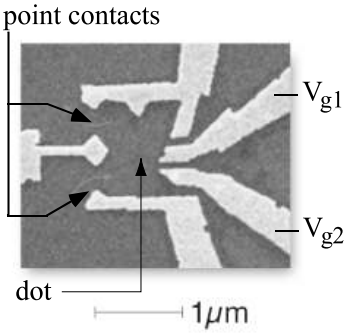


Figure 1.1: Micrograph of a quantum dot, taken from [15]. Here V_{g_1} and V_{g_2} are gate voltages used to control the size and number of electrons on the dot. To drive the system out of equilibrium we will later add a bias difference over the point contacts, that are connected to the leads.

quantum states of a quantum dot consisting of many electrons. We will start by analysing how non-interacting electrons distribute over the quantum dot. Each level, k, is either empty or filled, with occupancy $n_k = 0, 1$: all other occupancies are forbidden by the Pauli principle. A many-electron state is now described by the set $\{n_k\}$, and the energy of many particle the state is $E = \sum_k E_k n_k$ where E_k is the energy of each level. In the ground state the electrons occupy the lowest available energy levels. To create excitations in a dot with a fixed number of electrons we take one electron in state k and move it to an empty level k'. This gives an excitation energy $E_s = E_{k'} - E_k$. In the first generation we have one electron-hole pair excitation, in the second generation we add one more particle-hole excitation, and so on. If the lowest excitation has excitation energy δ_s, a basic question is how fast the number of states grows, when we allow a certain excitation energy E? The answer to this question is that this number grows exponentially. So far we have only considered non-interacting systems and the next question is of course what happens when we allow interactions? For a system with weak interactions the answer is not much. The reason for this is that there are still discrete states when interactions are taken into account, and the rules for occupation is essentially the same as for non-interacting systems. So at an increasing number of levels the excitation spectrum becomes continuous, and the dot is nothing but an isolated piece of metal. We know from Fermi liquid theory that the interaction effects can be disregarded and

we can expect the same behaviour as for a non interacting system. To see when interactions do play a role [13, 14] let us start from a formal Hamiltonian where the interactions are taken into account and spin degeneracy is assumed,

$$\mathcal{H} = \sum_{i\sigma} E_i c_{i\sigma}^\dagger c_{i\sigma} + \sum_{ijlm} \sum_{\sigma\sigma'} c_{i\sigma}^\dagger c_{j\sigma} U_{ijlm} c_{l\sigma'}^\dagger c_{m\sigma'}, \tag{1.5.1}$$

where i, j, l and m are orbital levels of the dot and $c_{i\sigma}$ is the electron annihilations operator in the level i with spin σ. The first sum gives the contribution of the non-interacting electrons and E_i is the energy of level i. The second sum gives the contribution from the Coulomb interaction and U_{ijlm} is the matrix element of the interaction $U(r_1 - r_2)$:

$$U_{ijlm} = \int dr_1 dr_2 \psi(r_1)\psi^*(r_1)U(|r_1 - r_2|)\psi(r_2)\psi^*(r_2), \tag{1.5.2}$$

where $\psi(r_i)$ are the wave functions of the corresponding levels. To simplify the interaction we assume that the energy spectrum consists of a randomly spaced set of levels with a mean level spacing Δ, which is small compared to all other relevant energy scales. We start by neglecting all the off-diagonal terms of U, which is a good approximation if the dimensionless conductance $g = E_T/\Delta \approx \sqrt{N} \to \infty$ [14, 16], where E_T is the Thouless energy. Neglecting the off-diagonal terms means that we have three types of interactions left: spin interaction, Cooper interactions and finally charging effects. Of these the first two can be neglected if the level spacing is larger than the energy of these interactions. Then the most important interaction is the charging energy. It provides the dominant energy difference between states that differ in the number of particles. Hence the interacting part of the Hamiltonian now takes the form

$$\mathcal{H}_{int} = \frac{1}{2} E_c \hat{N}^2, \tag{1.5.3}$$

where $\hat{N}$ is the total number operater and $E_c = e^2/2C$ is the charging energy of the dot from the total capacitance C. In term of eq. (1.5.2) we can write E_c in the form

$$E_c \sim \frac{1}{2}\int \frac{d^2r}{L^2} V(r), \tag{1.5.4}$$

where $V(r)$ is the Coulomb interaction. We now recover the expression in terms of the capacitance if we insert the Coulomb potential into the integral. The total Hamiltonian of the system is now

$$\mathcal{H} - \mu\hat{N} = \sum_i E_i c_i^\dagger c_i + \frac{E_c}{2}\hat{N}^2, \tag{1.5.5}$$

where μ is the chemical potential.

1.5.1 Coulomb blockade

The tunneling of one electron onto the dot is usually blocked by the classical Coulomb repulsion of electrons already on the dot, and the conductance is exponentially suppressed. This phenomenon is known as Coulomb blockade. By tuning the gate voltage, V_g, we can reach the situation where the energy of having N or $N+1$ electrons on the dot is degenerate. At the degeneracy point, the charge of the dot will fluctuate between N and $N+1$, and at this point we will obtain a finite conductance. In Fig. 1.2 we see the dependence of the conductance on the gate voltage. Each peak corresponds to a degeneracy point. At sufficiently low temperature these peaks will be spaced almost uniformly in the gate voltage, V_g, by an amount essentially proportional to the charging energy, E_c.

Coulomb blockade was first observed in tunneling junctions containing a small metallic particle which corresponds to the classical regime $\Delta \ll T \ll E_C$. In this regime where the tunneling occurs through a large number of levels, ($\sim T/\Delta$), a transport theory was introduced by Kulik and Shekhter [17] and further developed by Averin and Likharev [18].

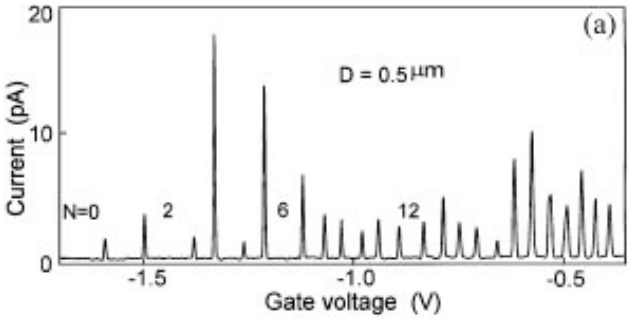


Figure 1.2: Conductance peaks in a Coulomb blockaded quantum dot, taken from [20]

In this thesis we are mainly interested in the quantum Coulomb blockade regime, which today is accessible in low temperature experiments in semiconductor quantum dots. In the quantum regime the tunneling takes place through a single resonance in the dot, see Fig. 1.3, which requires a different treatment.

Using Fermi's golden rule we will now calculate the conductance [19] for the quantum dot attached to non-interacting leads is described by a Hamiltonian

$$\mathcal{H} = \mathcal{H}_{lead} + \mathcal{H}_{dot} + \mathcal{H}_t, \tag{1.5.6}$$

where $\mathcal{H}_{dot}$ describes the isolated quantum dot

$$\sum_n \varepsilon_n d_n^\dagger d_n + \frac{E_C}{2}(\hat{N} - N_g)^2. \tag{1.5.7}$$

Where eN_g is the neutralising background charge (governed by the gate voltage, V_g, for the standard quantum dot), ε_n is the energy of the levels on the dot and d_n the associated annihilation operator. The tunneling and the leads are described by

$$\mathcal{H}_t = \sum_{\alpha k n} \left(t_{\alpha n} c_{\alpha k}^\dagger d_n + h.c. \right), \tag{1.5.8}$$

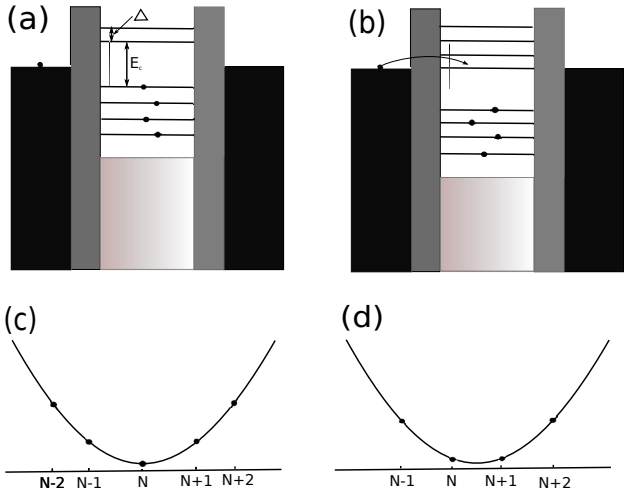


Figure 1.3: On the left we have the situation in a conductance valley, with N electrons on the dot. The cost of adding or removing one electron from the dot is very large. On the right we have the situation around one of the peaks of conductance, see Fig. 1.2, where having N or $N+1$ electrons on the dot is energetically degenerate. The electrons can now easily tunnel through the dot.

and

$$\mathcal{H}_{lead} = \sum_{\alpha k} \xi_k c^\dagger_{\alpha k} c_{\alpha k}, \tag{1.5.9}$$

respectively, where $t_{\alpha n}$ is the tunneling coefficiaent, $\xi_k = \varepsilon_n - \mu$, and $\alpha = L, R$ signifies the left or right lead. We will now calculate the conductance near the degeneracy point [19] , where N_g is a half integer, so that

$$|E_N - E_{N+1}| < T, \tag{1.5.10}$$

where E_N is the charging energy of a state with N electrons. This will be calculated in the region

$$\Delta \ll T \ll E_c, \tag{1.5.11}$$

where E_c is the charging energy, and Δ the level spacing. The condition in eq. (1.5.10) means that we only need to take two charge states into consideration and eq. (1.5.11) enables us to treat the discrete set of particle states as a continuum with the density of states $1/\Delta$. We also assume that we are in the sequential tunneling regime where transitions between the discrete levels happen before the electron has the chance to escape into the dot. This means that tunneling events between the two junctions are independent. Using Fermi's golden rule the current can now be written as

$$\begin{aligned} I_\alpha &= e\frac{2\pi}{\hbar}\sum_{kn}|t^2_{\alpha n}|\delta(\xi_k + eV_\alpha - \varepsilon_n + E_N - E_{N+1}) \\ &\quad \times\{P_N f(\xi_k)(1-f(\varepsilon_n)) - P_{N+1}f(\varepsilon_n)(1-f(\xi_k))\}, \end{aligned} \tag{1.5.12}$$

where P_N is the probability to find N electrons on the dot, V is the applied bias voltage and f is the Fermi distribution. If we replace the summation over k and n with integrals and integrate over the corresponding continuum we obtain the following expression

$$I_\alpha = \frac{4e}{\hbar}\frac{\Gamma_\alpha}{\Delta}\left(P_N F(E_N - E_{N+1} - eV_\alpha) - P_{N+1}F(eV_\alpha + E_{N+1} - E_N\right), \tag{1.5.13}$$

where $\Gamma_1 = \pi\nu_0|t_L|^2$ and $\Gamma_2 = \pi\nu_0|t_R|^2$ with ν_0 is the mean Tunneling density of states for non-interacting electrons and

$$F(\omega) = \frac{\omega}{e^{\omega/T}-1}. \tag{1.5.14}$$

In the Coulomb blockade regime the dot cannot acumulate charge and therefore current conservation holds,

$$I = I_L = -I_R. \tag{1.5.15}$$

where $I_{L/R}$ is the current through the left/right contact, respectively. Combining eqs. (1.5.13) and (1.5.15) and the obvious normalisation condition $P_N + P_{N+1} = 1$ we can calculate the current through the system in response to the applied bias voltage $V = V_L - V_R$. In our system we add the bias voltage symmetrically so the bias voltage in the left, V_L, and right lead, V_R, have the same magnitude but different size, i.e. $V_L = -V_R = V/2$. This give us the linear conductance through the dot,

$$G = \lim_{V \to 0} \frac{dI}{dV} = \frac{4\pi}{\hbar} \frac{\Gamma_1 \Gamma_2}{\Gamma} \frac{(E_N - E_{N+1})/T}{\sinh\left(2(E_N - E_{N+1})\right)/T}, \tag{1.5.16}$$

where $\Gamma = \Gamma_1 + \Gamma_2$ In chapter 3 we reproduce this result of the conductance with the help of tunneling density of states. A gap in the tunneling density of states corresponds to a suppression of the conductance, since there are no states to tunnel into. Around the degeneracy point this gap is starting to close, and this corresponds to a finite conductance.

1.6 Kondo effect

In 1936 de Haas and van den Berg [21] found a resistance minimum as a function of temperature in a disordered metal with a finite concentration of magnetic impurities. This was very surprising since theory stated that resistance in metals was due to phonon and impurity scattering. While impurity scattering is strongly temperature independent we know that phonon scattering is temperature dependent and can be estimated by the Debye model. At low temperatures the phonon scattering is suppressed and it vanishes at zero temperature. Later experiments showed that the effect had to depend on magnetic impurities since changing this concentration made the effect more or less pronounced.

A step forward to understand the phenomena was done when Anderson introduced an impurity model [22] that was designed to model isolated magnetic impurities in a non-magnetic environment. Using this model Anderson succeeded in estimating a parameter range where

such isolated impurities could exist in a metallic environment. The big breakthrough came in 1964 when Kondo [23] did perturbation theory on a model of a localised magnetic moment represented by an isolated spin which interacts with the collective spin of the conduction electrons via a Heisenberg coupling. By performing perturbation theory in the coupling strength J, he found that for low temperatures the scattering amplitude for electrons increases sharply towards the Fermi energy. This increases the electron scattering, which in a metal increases the resistance. Kondo also succeed in explaining the connection to the impurity concentration and obtained agreement with the experimental results.

In the original solution there is a logarithmic divergence in the perturbation theory at $T = 0$. This problem turned out to be very hard to solve. The first progress towards the solution was done by Abrikosov who identified the most divergent diagrams of each order of the perturbation theory and did a resummation of these terms. This solved the problem for the ferromagnetic case, $J > 0$, but for the anti-ferromagnetic case the divergence now occurred at a finite temperature, T_K, which is known as the Kondo temperature.

The explanation for the failure of perturbation theory was provided by Anderson in his poor man's scaling approach [24]. Anderson reduced the bandwith cut-off in J and found that this increased the coupling strength for the anti-feromagnetic case. Roughly speaking this means that the low temperature physics is dominated by virtual scattering processes and these processes get more and more profound when $T \to 0$. From this analysis it is clear that perturbation theory in J is doomed in this limit.

Wilson refined this result using a numerical approach [25]. Nozières derived an effective low energy theory [26] that reproduces the results of Wilson. He used a simple scattering description close to the strong coupling fixed point of the Kondo model and obtained an effective Fermi liquid theory.

Finally a non-pertubative solution of the problem was derived using the Bethe ansatz by Andrei [27], Wiegman and Tsvelik [28, 29, 30]. From the Bethe ansatz solution it is possible to extract thermodynamic properties such as the magnetic susceptibility and heat capacity. The downside is that it is not possible to extract dynamic or non-equilibrium properties. The Kondo effect was predicted theoretically in quantum dots in 1988 [31, 32] and verified experimentally 10 years later [3, 33]. The Kondo effect is very different in quantum dots and metals. The reason for this is that in a metal, electron scattering lowers the conductivity. Hence the Kondo result explains why we obtain an increase in the resistivity. In a quantum dot the only accessible transport mechanism is electron scattering. Hence in a quantum dot the increased scattering leads to an increase in the conductivity.

Chapter 2

GREEN FUNCTIONS

An exact solution of a quantum field theory corresponds to knowing all the correlation functions of the field variables. A correlation function measures the overlap between our initial state and a state where we insert a particle or a hole at a specific time t and place x and try to extract it at a later time t' and position x'. With the knowledge of all the correlation functions of a system we know how the correlations in the system will evolve in time.

There are three different main types of Green functions that are useful for different problems in condensed matter theory. For zero temperature we have zero temperature Green functions [34], for finite temperature in equilibrium we have Matsubara Green functions [34] and for a non-equilibrium system at finite temperature we have Keldysh Green functions [35, 36, 37, 38]. Keldysh Green functions of course work also for equilibrium and zero temperature problems but their algebraic structure is slightly more complicated then the other two Green functions which, often makes them unsuitable for equilibrium problems.

We will in this chapter introduce the different types of Green functions, but the emphasis in the remaining part of this thesis will be on non-equilibrium systems, so we will concentrate on the Keldysh formulation.

2.1 Green functions: the basics

We start from the basics with the definition of the single particle zero temperature time ordered Green function, also known as the causal Green function,

$$G(x,t;x',t') = -i\frac{{}_0\langle\psi_0|T\hat{\psi}_H(x,t)\hat{\psi}_H^\dagger(x',t')|\psi_0\rangle_0}{{}_0\langle\psi_0|\psi_0\rangle_0}. \tag{2.1.1}$$

Here $|\psi_0\rangle_0$ is the ground state in the interacting system and T is the time ordering operator. The time ordering operator always moves the operator with the earlier time to the right

$$TA(t)B(t') = \theta(t-t')A(t)B(t') \pm \theta(t'-t)B(t')A(t), \tag{2.1.2}$$

where $+/-$ corresponds to bosons/fermions. The operator $\hat{\psi}_H(t)$ is in the Heisenberg representation, and so its time dependence has the form

$$\hat{\psi}_H(t) = e^{iHt}\hat{\psi}(t=0)e^{-iHt}. \tag{2.1.3}$$

This definition can be extended to equilibrium systems at finite temperature

$$G(x,t;x',t') = -iTr\left(\rho T\hat{\psi}_H(x,t)\hat{\psi}_H^\dagger(x',t')\right)., \tag{2.1.4}$$

where ρ is the density matrix operator and Tr is the sum over all diagonal elements. Now that we have introduced the Green function in an equilibrium system it is time to show that it is useful for calculating real physical quantitites such as the particle density

$$\langle n(x)\rangle = \langle\psi^\dagger(x)\psi(x)\rangle. \tag{2.1.5}$$

This is equivalent to eq. (2.1.4) if t is before t'. So we can now define the particle density as

$$\langle n(x)\rangle = -iG(x,t;x,t^+), \tag{2.1.6}$$

where

$$t^+ = \lim_{\varepsilon\to 0}(t+\varepsilon). \tag{2.1.7}$$

This fixes the time ordering since t^+ is always infinitesimally larger then t. In an homogenous system the Green functions always depend only on the difference in time and space and not on each variable individually. So we write the Green function as

$$G(x,t;x',t') = G(x-x',t-t'). \tag{2.1.8}$$

In this case it is more convenient to work in Fourier space

$$G(k,\omega) = \int d^3x \int dt e^{i\omega(t-t')} e^{-ik(x-x')} G(x-x',t-t'). \tag{2.1.9}$$

We will now introduce 4 additional Green functions, the retarded, advanced, lesser than and greater than:

$$\begin{aligned}
G^R(x,t;x',t') &= -i\theta(t-t')\langle\{\hat{\psi}(x,t),\hat{\psi}^\dagger(x',t')\}\rangle, && (2.1.10)\\
G^A(x,t;x',t') &= i\theta(t'-t)\langle\{\hat{\psi}(x,t),\hat{\psi}^\dagger(x',t')\}\rangle, && (2.1.11)\\
G^<(x,t;x',t') &= i\langle\hat{\psi}^\dagger(x',t')\hat{\psi}(x,t)\rangle, && (2.1.12)\\
G^>(x,t;x',t') &= -i\langle\hat{\psi}(x,t)\hat{\psi}^\dagger(x',t')\rangle. && (2.1.13)
\end{aligned}$$

where $\{.,.\}$ is the anti-commutator. The four Green functions are not independent and can easily be seen to fulfil the relation

$$G^R - G^A = G^> - G^<. \tag{2.1.14}$$

The reason we bother with introducing all four of them is that they have different properties and are useful in different situations. $G^{R,A}$ have a nice analytic structure and are well suited to calculate physical response. Information about spectral properties, density of states and scattering rates are all contained in the retarded and advanced Green functions. The lesser and greater Green functions are linked to physical variables such as particle densities and currents. The benefit of the time ordered Green function, G, is that is has a systematic perturbation theory. We note that the time ordered Green function can be written in terms or lesser and greater Green functions

$$G(x,t;x',t') = \theta(t-t')G^>(x,t;x',t') + \theta(t'-t)G^<(x,t;x',t'). \tag{2.1.15}$$

Also the retarded and advanced Green functions be can expressed in terms of lesser and greater Green functions:

$$G^{R,A}(x,t;x',t') = \pm\theta(\pm t \mp t')\left(G^>(x,t;x',t') - G^<(x,t;x',t')\right). \tag{2.1.16}$$

where the upper/lower sign in in eq. (2.1.16) corresponds to the retarded/advanced Green function, respectively.

2.2 Keldysh Green functions

The building block of constructing a perturbation theory in an equilibrium system is that both the initial and final states are the same. In a non-equilibrium system this can no longer

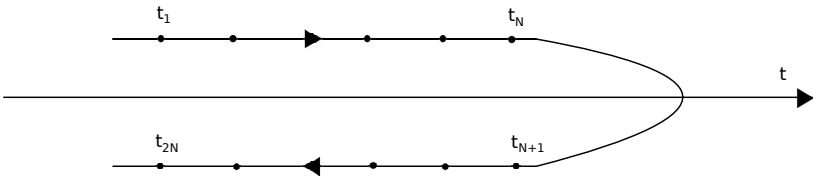


Figure 2.1: The standard closed Keldysh time contour. The upper/lower branch is the forward/backward branch

be assumed. To construct a non-equilibrium theory one therefore needs to avoid reference to asymptotically large times. In the Keldysh technique [35, 36, 37] we start at a distant past and evolve the system forward to a time t and and then evolve the time backwards so that we end up at our initial state. In this way we will construct a more abstract time evolution since we can have evolution along both the forward part , ψ_+, and backward part, ψ_-, of the time contour, see Fig. 2.1. We can now introduce contour ordered Green function with matrix structure

$$G(t,t') = \begin{pmatrix} G^{++}(t,t') & G^{+-}(t,t') \\ G^{-+}(t,t') & G^{--}(t,t') \end{pmatrix}, \tag{2.2.1}$$

where

$$G^{<}(t,t') = G^{+-}(t,t') = -i\langle\hat{\psi}_+(t)\hat{\psi}_-^\dagger(t')\rangle, \tag{2.2.2}$$

$$G^{>}(t,t') = G^{-+}(t,t') = -i\langle\hat{\psi}_-(t)\hat{\psi}_+^\dagger(t')\rangle,$$

$$G^{T}(t,t') = G^{++}(t,t') = -i\langle T\hat{\psi}_+(t)\hat{\psi}_+^\dagger(t')\rangle, \tag{2.2.3}$$

$$G^{\bar{T}}(t,t') = G^{--}(t,t') = -i\langle \bar{T}\hat{\psi}_-(t)\hat{\psi}_-^\dagger(t')\rangle. \tag{2.2.4}$$

where $\bar{T}$ denotes anti-time ordering. If we now note that

$$G^{T}(t,t') = \theta(t-t')G^{>}(t,t') + \theta(t'-t)G^{<}(t,t'), \tag{2.2.5}$$

$$G^{\bar{T}}(t,t') = \theta(t-t')G^{<}(t,t') + \theta(t'-t)G^{>}(t,t'), \tag{2.2.6}$$

we obtain the relation

$$G^T + G^{\bar{T}} = G^> + G^<. \tag{2.2.7}$$

So the four components are not independent. This redundancy can be removed by performing the Keldysh rotation to the Larkin-Ovchinnikov basis [39]:

$$\begin{aligned} \hat{\psi}_1(t) &= \frac{1}{\sqrt{2}}(\hat{\psi}_+(t) + \hat{\psi}_-(t)) \qquad \hat{\psi}_2(t) = \frac{1}{\sqrt{2}}(\hat{\psi}_+(t) - \hat{\psi}_-(t)), \\ \hat{\psi}_1^\dagger(t) &= \frac{1}{\sqrt{2}}(\hat{\psi}_+^\dagger(t) - \hat{\psi}_-^\dagger(t)) \qquad \hat{\psi}_2^\dagger(t) = \frac{1}{\sqrt{2}}(\hat{\psi}_+^\dagger(t) + \hat{\psi}_-^\dagger(t)). \end{aligned} \tag{2.2.8}$$

Performing the Keldysh rotation eq. (2.2.8) on eq. (2.2.4) we obtain

$$G = \begin{pmatrix} G^R(t,t') & G^K(t,t') \\ 0 & G^A(t,t') \end{pmatrix}, \tag{2.2.9}$$

where

$$G^R = \frac{1}{2}\left(G^T + G^< - G^> - G^{\bar{T}}\right) = \theta(t-t')\left(G^<(t,t') - G^>(t,t')\right), \tag{2.2.10}$$

$$G^A = \frac{1}{2}\left(G^T - G^< + G^> - G^{\bar{T}}\right) = -\theta(t'-t)\left(G^<(t,t') - G^>(t,t')\right), \tag{2.2.11}$$

$$G^K = G^<(t,t') + G^>(t,t'). \tag{2.2.12}$$

The form of the Green function in eq. (2.2.9) can also be obtained directly from eq. (2.2.1) by performing the operation [37, 39]

$$G = L\sigma_z G L, \quad \text{where} \quad L = \frac{1}{\sqrt{2}}\left(\mathbb{I} - i\sigma_y\right), \tag{2.2.13}$$

and using the relation eq. (2.2.7). In equilibrium the situation is simplified and we can find a relation between the three Green functions G^R, G^A and G^K. To do this we make the observation that in equilibrium the Hamiltonian does not dependent on t directly so all our

Green functions depend on $t-t'$ only, rather than on both times independently. If we now take the Fourier transform with respect to $t-t'$ we obtain

$$i\left(G^R(\omega)-G^A(\omega)\right) = \int_{-\infty}^{\infty} dt e^{i\omega t}\mathrm{Tr}\left(e^{-\beta H}\left\{e^{iHt}\hat{\psi}e^{-iHt},\hat{\psi}^\dagger\right\}\right), \quad (2.2.14)$$

$$iG^K(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t}\mathrm{Tr}\left(e^{-\beta H}\left[e^{iHt}\hat{\psi}e^{-iHt},\hat{\psi}^\dagger\right]\right). \quad (2.2.15)$$

We now take a closer look at eq. (2.2.14), by splitting the integrals into two parts $I_1 = \int_{-\infty}^{\infty} dt e^{i\omega t}\mathrm{Tr}\left(e^{-\beta H}e^{iHt}\hat{\psi}e^{-iHt}\hat{\psi}^\dagger\right)$ and $I_2 = \int_{-\infty}^{\infty} dt e^{i\omega t}\mathrm{Tr}\left(e^{-\beta H}\hat{\psi}^\dagger e^{iHt}\hat{\psi}e^{-iHt}\right)$. We start with I_1 and make the substitution $t \to t - i\beta$ and use the property $tr(XY) = tr(YX)$ of the trace to obtain

$$\begin{aligned}\int_{-\infty}^{\infty} dt e^{i\omega t}\mathrm{Tr}\left(e^{-\beta H}e^{iHt}\hat{\psi}e^{-iHt}\hat{\psi}^\dagger\right) &= e^{\beta\omega}\int_{-\infty}^{\infty} dt e^{i\omega t} tr\left(e^{iHt}\hat{\psi}e^{-iHt}\hat{\psi}^\dagger\right) \\ &= e^{\beta\omega}\int_{-\infty}^{\infty} dt e^{i\omega t} tr\left(e^{-\beta H}a^\dagger e^{iHt}ae^{-iHT}\right). \quad (2.2.16)\end{aligned}$$

We see that eq. (2.2.16) is equivalent to the statement $I_1 = e^{\beta\omega}I_2$. By using this relation we obtain

$$\frac{I_1 - I_2}{I_1 + I_2} = \frac{e^{\beta\omega}-1}{e^{\beta\omega}+1} = \tanh\left(\frac{\beta\omega}{2}\right). \quad (2.2.17)$$

Using eqs. (2.2.14) and (2.2.15) we can rewrite eq. (2.2.17) as

$$G^K(\omega) = \tanh\left(\frac{\beta\omega}{2}\right)\left(G^R(\omega)-G^A(\omega)\right). \quad (2.2.18)$$

In general we can write the Keldysh component:

$$G^K = G^R\cdot F - F\cdot G^A, \quad (2.2.19)$$

where F is a Hermitian matrix that can be found from a quantum kinetic equation [36, 38].

2.3 Functional field integrals

Quantum field theory can be formulated in two different ways: the formalism of canonically quantised field operators [34] and functional integration. Functional path integrals provide an entire spectrum of novel routes toward approaches to quantum mechanical problems (controlled semi-classical limits, analogies to classical mechanics, statistical mechanics, concepts of topology and geometry, etc.), for a review of this topic see the recent book by Altland and Simons [40]. In this section our goal is to construct the many-body path integral [40]. The basic idea is to segment the time evolution of our system into infinitesimal time slices and absorb as much of the quantum dynamical phase accumulated during the short-time propagation into a set of suitably chosen eigenstates.

2.3.1 Coherent states

A coherent state is an eigenstate of the annihilation operator

$$\hat{a}_i|\eta\rangle = \eta_i|\eta\rangle, \tag{2.3.1}$$

where η_i is the eigenvalue of $\hat{a}_i$. We can write the general state

$$|\eta\rangle = \exp\left(\sum_i \eta_i \hat{a}_i^\dagger\right)|0\rangle. \tag{2.3.2}$$

One thing that complicates things at this stage is the anticommuting properties of the fermionic operators $\{\hat{a}_i, \hat{a}_j\} = 0$ if $i \neq j$, since this implies that the eigenvalues of the coherent states also have to anti-commute

$$\eta_i\eta_j = -\eta_j\eta_i. \tag{2.3.3}$$

Clearly these objects, called Grassmannn variables, cannot be normal numbers. To be able to define the coherent states we first have to define the Grassmann numbers and study their properties more carefully. Grassmann numbers are elements of the Grassmann algebra which consists of elements that all anti-commute. This condition implies that the square of a Grassmann number is zero since

$$\{\eta_i, \eta_i\} = \eta_i^2 + \eta_i^2 = 0. \tag{2.3.4}$$

Functions of Grassmann numbers are defined via their Taylor expansion

$$f(\xi_1, \xi_2, \cdots, \xi_k) = \sum_{n=0}^{\infty} \sum_{i_1,\cdots,i_n=1}^{k} \frac{1}{n!} \frac{\partial^n f}{\partial \xi_{i_1} \cdots \partial_{i_n}}\bigg|_{\xi=0} \xi_{i_1} \cdots \xi_{i_n}, \qquad \xi_{i_1}, \cdots \xi_{i_k} \in \mathcal{A}, \tag{2.3.5}$$

where f is an analytic function and $\mathcal{A}$ is the Grassman algebra. From eq. (2.3.4) it follows that any function of a Grassmann number is must be linear. So in the one variable case

$$f(\eta) = f(0) + f'(0)\eta. \tag{2.3.6}$$

Now let us turn to differentiation of Grassman numbers, which is defined by

$$\partial_{\eta_i} \eta_j = \delta_{ij}. \tag{2.3.7}$$

If this definition is to be consistent with the anti-commutation relations, the derivative has to be anti-commutative itself. In particular $\partial_{\eta_i} \eta_j \eta_i = -\eta_j$ if $i \neq j$. Finally we have integration over Grassmann fields [40]

$$\int d\eta_i = 0, \qquad \int d\eta_i \eta_i = 1. \tag{2.3.8}$$

If we take a closer look at eqs. (2.3.5)-(2.3.8) we observe that the action of Grassmann integration and differentiation is the same,

$$\int d\eta f(\eta) = \int d\eta (f(0) + f'(0)\eta) = f'(0) = \partial_\eta(\eta). \tag{2.3.9}$$

The Grassman version of the Gaussian integral $\int d\eta d\bar{\eta} e^{-\bar{\eta}\eta} = 1$ does not contain the factors of π of the standard Gaussian integrals. We end the introduction of coherent states with some useful indentities

$$\int d\eta d\bar{\eta} e^{-\bar{\eta} a \eta} = a, \tag{2.3.10}$$

and the multidimensional generalisation of the Gaussian integral

$$\int d(\bar{\phi}, \phi) e^{\bar{\phi}^T A \phi} = \det A. \tag{2.3.11}$$

2.3.2 The functional integral Green function

In the two coming chapters we will work in the functional integral approach, so we will need to define Green functions within this approach. This will be done in this section using the knowledge we have just gained about Grassmann numbers. Let us start from the standard definition of a Green function

$$G(t, t') = \frac{1}{\mathcal{Z}} \sum_n \langle n | T_c \hat{\psi} \hat{\psi}^\dagger e^{i \int_C H(t) dt} | n \rangle, \tag{2.3.12}$$

where the time integral is over the Kelsdysh contour C in Fig. 2.1. Now we insert the resolution of identity

$$\hat{1} = \int \Pi_n d\bar{\eta}_n d\eta_n e^{\sum_n \bar{\eta}_n \eta_n} |\eta\rangle\langle\eta|, \tag{2.3.13}$$

into the definition in eq. (2.3.12)

$$G(t,t') = \frac{1}{\mathcal{Z}} \int \Pi_m d\bar{\eta}_m d\eta_m e^{\sum_m \bar{\eta}_m \eta_m} \sum_n \langle n|T_c \hat{\psi}\hat{\psi}^\dagger e^{i\int_C H(t)dt}|\eta\rangle\langle\eta|n\rangle. \tag{2.3.14}$$

The next step is to get rid of the sum over n by swapping the order $\langle n|T_c \hat{\psi}\hat{\psi}^\dagger e^{i\int_C H(t)dt}|\eta\rangle\langle\eta|n\rangle$ and using that

$$\sum_n |n\rangle\langle n| = \hat{1}. \tag{2.3.15}$$

Since there are elements of anti-commuting Grassmann variables

$$\langle n|T_c \hat{\psi}\hat{\psi}^\dagger e^{i\int_C H(t)dt}|\eta\rangle\langle\eta|n\rangle \neq \langle\eta|n\rangle\langle n|T_c \hat{\psi}\hat{\psi}^\dagger e^{i\int_C H(t)dt}|\eta\rangle, \tag{2.3.16}$$

and we have to be a bit more careful. We start with calculating the overlap $\langle\eta^{(0)}|n\rangle$. Using the definitions eq. (2.3.2) and defining the state

$$|n\rangle = \hat{a}^\dagger_{p_1}\hat{a}^\dagger_{p_2}\cdots\hat{a}^\dagger_{p_n}|0\rangle \tag{2.3.17}$$

we can write the overlap

$$\begin{aligned}
\langle\bar{\eta}^{(0)}|n\rangle &= \langle 0|e^{-\sum_p \hat{a}_p\bar{\eta}_p}\hat{a}^\dagger_{p_1}\hat{a}^\dagger_{p_2}\cdots\hat{a}^\dagger_{p_n}|0\rangle \\
&= \langle 0|(-\hat{a}_{p_1}\bar{\eta}^{(0)}_{p_1})\cdots(-\hat{a}_{p_n}\bar{\eta}^{(0)}_{p_n})\hat{a}^\dagger_{p_1}\hat{a}^\dagger_{p_2}\cdots\hat{a}^\dagger_{p_n}|0\rangle \\
&= (-1)^n(-1)^{n(n-1)/2}\bar{\eta}_{p_1}\cdots\bar{\eta}_{p_n}\langle 0|\hat{a}_{p_1}\cdots\hat{a}_{p_n}\hat{a}^\dagger_{p_1}\cdots\hat{a}^\dagger_{p_n}|0\rangle \\
&= (-1)^n(-1)^{n(n-1)/2}\bar{\eta}^{(0)}_{p_1}\cdots\bar{\eta}^{(0)}_{p_n}.
\end{aligned} \tag{2.3.18}$$

So when we want to pull eq. (2.3.18) through the expression

$$\langle n|T_c \hat{\psi}\hat{\psi}^\dagger e^{i\int_C H(t)dt}|\eta\rangle, \tag{2.3.19}$$

we see that the expansions of all parts contain pairs of elements that anti-commute with the η_{p_i} terms with one exception $\langle n|$, which has n annihilation operators which each contribute a $(-1)^n$ when we pull then η_i through. So in total we get a sign of $(-1)^{n^2}$. We obtain

$$\begin{aligned} G(t,t') &= \frac{1}{\mathcal{Z}}\int \Pi_m d\bar{\eta}_m d\eta_m e^{\sum_m \bar{\eta}_m\eta_m}\sum_n \langle n|T_c\hat{\psi}\hat{\psi}^\dagger e^{i\int_C H(t)dt}|\eta\rangle\langle\eta|n\rangle \\ &= \frac{1}{\mathcal{Z}}\int \Pi_m d\bar{\eta}_m d\eta_m e^{\sum_m \bar{\eta}_m\eta_m}\sum_n (-1)^n\langle\eta|n\rangle\langle n|T_c\hat{\psi}\hat{\psi}^\dagger e^{i\int_C H(t)dt}|\eta\rangle \\ &= \frac{1}{\mathcal{Z}}\int \Pi_m d\bar{\eta}_m d\eta_m e^{\sum_m \bar{\eta}_m\eta_m}\langle -\eta|T_c\hat{\psi}\hat{\psi}^\dagger e^{i\int_C H(t)dt}|\eta\rangle. \end{aligned} \tag{2.3.20}$$

We now split the time contour into N parts with the width δ_i

$$G(t,t') = -\frac{i}{\mathcal{Z}}\int \mathcal{D}\bar{\eta}\mathcal{D}\eta e^{\bar{\eta}\eta}\langle -\eta|T_c\hat{\psi}\hat{\psi}e^{-i\sum_i \delta_i H}|\eta\rangle. \tag{2.3.21}$$

We now split the exponential into N parts and insert a resolution of identity in between each term,

$$\begin{aligned} G(t,t') &= -\frac{i}{\mathcal{Z}}\int \mathcal{D}^{N+1}\psi\mathcal{D}^{N+1}\bar{\psi}e^{\bar{\psi}_0\psi_0-\sum_{i=1}^N \bar{\psi}_i\psi_i} \\ &\quad \langle\psi_{N+1}|e^{i\delta_i H}|\psi_N\rangle\langle\psi_N|e^{i\delta_i H}|\psi_{N-1}\rangle\cdots\langle\psi_{n+m+1}|e^{i\delta_i H}\hat{\psi}|\psi_{n+m}\rangle \\ &\quad \langle\psi_{n+m}|e^{i\delta_i H}|\psi_{n+m-1}\rangle\cdots\langle\psi_n|e^{i\delta_i H}|\psi_N\rangle\langle\psi_{m+1}|\hat{\psi}^\dagger e^{i\delta_i H}|\psi_m\rangle \\ &\quad \langle\psi_m|e^{i\delta_i H}|\psi_{m-1}\rangle\cdots\langle\psi_1|e^{i\delta_i H}|\psi_0\rangle. \end{aligned} \tag{2.3.22}$$

If we demand that the Hamiltonian is normal ordered we can now write the element

$$\langle\psi_{n+1}|e^{i\delta H}|\psi_n\rangle = \langle\psi_{n+1}|1-i\delta_i H|\psi_n\rangle = e^{i\delta H(\bar{\psi}_{n+1},\psi_n)}e^{\bar{\psi}_{n+1}\psi_n}. \tag{2.3.23}$$

We can now write the Keldysh Green function in the functional integral representation

$$G(t,t') = \frac{i}{\mathcal{Z}}\int \mathcal{D}^{N+1}\psi\mathcal{D}^{N+1}\bar{\psi}\psi\bar{\psi}e^{iS}, \tag{2.3.24}$$

where

$$iS = \sum_{i=0}^{N} \delta_i \left\{ \bar{\psi}_{i+1} \frac{\psi_i - \psi_{i+1}}{\delta_i} - i\mathcal{H}(\bar{\psi}_{i+1}, \psi) \right\} \tag{2.3.25}$$

$$= i \int_C dt \left\{ \bar{\psi} i\partial\psi - \mathcal{H}(\bar{\psi}(t), \psi(t)) \right\}. \tag{2.3.26}$$

2.4 Some matrix identities

In the previous section we derived the functional integral form of the Keldysh Green function. A typical functional integral takes the form

$$\int \mathcal{D}\psi\mathcal{D}\bar{\psi} e^{-\bar{\psi}_\alpha \mathbf{M}_{\alpha\beta}\psi_\beta}. \tag{2.4.1}$$

We know from the basic theory of Grassmann algebra that this takes the form

$$\int \mathcal{D}\psi\mathcal{D}\bar{\psi} e^{-\bar{\psi}_\alpha \mathbf{M}_{\alpha\beta}\psi_\beta} = \det \mathbf{M}. \tag{2.4.2}$$

We will in this section have a look at some special cases that will turn out to be useful in the later chapters. We assume that M is a Matrix of the form

$$M = \begin{pmatrix} -1 & 0 & 0 & \cdots & a_{N+1} \\ a_1 & 1 & 0 & \cdots & 0 \\ 0 & -a_2 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & -a_N & 1 \end{pmatrix}, \tag{2.4.3}$$

where the matrix element $a_i = 1 - i\delta_i\phi_i$, i labels the time and ϕ_i is the Hamiltonian at time t_i. Using eq. (2.4.2) we can write the path integrals as a function of the M matrix in the

form:

$$-\int \mathcal{D}\psi\mathcal{D}\bar{\psi}e^{-\bar{\psi}\mathbf{M}\psi} = -\det\mathbf{M} = 1 + \prod_{i=1}^{N+1} a_i, \tag{2.4.4}$$

$$-\int \mathcal{D}\psi\mathcal{D}\bar{\psi}e^{-\bar{\psi}\mathbf{M}\psi}\psi_k\bar{\psi}_l = \begin{cases} -\prod_{i=1}^{N+1} a_i & \text{if} \quad k > l \\ \prod_{i=1}^{k-1}\prod_{l}^{N+1} a_i & \text{if} \quad k < l \end{cases}. \tag{2.4.5}$$

Look at how the typical example where $a_i = 1 - i\delta_i\phi_i \approx e^{i\delta_i\phi_i}$. Inserting this relation into eqs. (2.4.4) and (2.4.5) we obtain the very usual path integral relations

$$\int \mathcal{D}\psi\mathcal{D}\bar{\psi}e^{-\bar{\psi}\mathbf{M}\psi} = 1 + e^{-i\int\phi(t)dt}, \tag{2.4.6}$$

$$-\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\psi(t)\bar{\psi}(t')e^{-\bar{\psi}\mathbf{M}\psi} = \begin{cases} e^{-i\int_{t'}^{t}\phi(t'')dt''} & \text{if} \quad t > t' \\ -e^{-i\int_{t_0}^{t}\phi(t'')dt''-i\int_{t'}^{t_{N+1}}\phi(t'')dt''} & \text{if} \quad t < t' \end{cases}. \tag{2.4.7}$$

2.4.1 Analytic continuation

In coming chapters we will encounter many terms with the structure

$$C(t,t') = \int_C dt_1 A(t,t_1)B(t_1,t'). \tag{2.4.8}$$

In this section we will derive rules about how we can write $C^<$ in terms of lesser, greater, advanced or retarded A and B components [38]. We will start to derive the identity for $C^<$. This means that we have fixed t on the forward branch and t' on the backward branch. The first step is to deform the standard contour in the way illustrated in Fig. 2.2 so that we can now write eq. (2.4.8) in the form

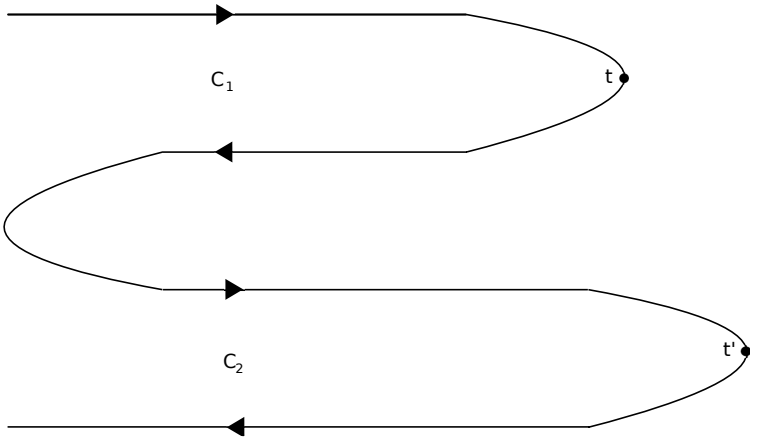


Figure 2.2: The deformed Keldysh time contour split in two parts C_1 and C_2. Time ordering is such that all times on C_1 are before any time on C_2.

$$C^<(t,t') = \int_{C_1} dt_1 A(t,t_1)B^<(t_1,t') + \int_{C_2} dt_1 A^<(t,t_1)B(t_1,t'). \tag{2.4.9}$$

Here we have used that if t_1 is on the C_1 contour then it is always less than t' in the contour sense. In the same way anything on contour C_2 is greater than t. Taking the first term in eq. (2.4.9) we can split the integration into two parts

$$\begin{aligned} C^<(t,t') &= \int_{-\infty}^{t} A^>(t,t_1)B^<(t_1,t') + \int_{t}^{-\infty} A^<(t,t_1)B^<(t_1,t') \\ &\equiv \int_{-\infty}^{\infty} A^R(t,t_1)B^<(t_1,t'), \end{aligned} \tag{2.4.10}$$

where we used the definition of G^R in eq. (2.1.16). Using a similar argument on the second term we obtain the following relation

$$C^<(t,t') = \int_{-\infty}^{\infty} dt_1 \left[A^R(t,t_1)B^<(t_1,t') + A^<(t,t_1)B^A(t_1,t')\right]. \tag{2.4.11}$$

It is easy to see that if we swap the order of the external times to obtain $C^>$ we just need to swap $<$ for $>$ in eq. (2.4.11) so that

$$C^>(t,t') = \int_{-\infty}^{\infty} dt_1 \left[A^R(t,t_1)B^>(t_1,t') + A^>(t,t_1)B^A(t_1,t')\right]. \tag{2.4.12}$$

For the case of products of three, i.e. $D = ABC$, we obtain [38]

$$D^< = A^R B^R C^< + A^R B^< C^A + A^< B^A C^A. \tag{2.4.13}$$

2.5 Noise in a resonant level model

We will close this chapter by calculating the noise in a resonant level model. This will enable us to use most of the theory introduced in this section. The key motivation for doing this calculation is that the result will be used when we study the noise in a Coulomb blockaded quantum dot. In this strongly interacting system we can rewrite part of the problem as a resonant level model and use the results from this section with small modifications. But more about this in chapter 4. We start by defining the model

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_t + \mathcal{H}_c, \tag{2.5.1}$$

where

$$\mathcal{H}_0 = \sum_{\alpha,k} \varepsilon_k c^\dagger_{\alpha k} c_{\alpha k}, \tag{2.5.2}$$

$$\mathcal{H}_t = \sum_{\alpha,k} \left(t_{\alpha k} c^\dagger_{\alpha k} d + h.c. \right), \tag{2.5.3}$$

$$\mathcal{H}_c = \varepsilon_n d^\dagger d. \tag{2.5.4}$$

where ε_n is the energy of the resonant level. We start from the current operator:

$$I_L = ie \sum_k \left[t_{Lk} \hat{c}^\dagger_{Lk} \hat{d} - t^*_k \hat{d}^\dagger \hat{c}_{Lk} \right]. \tag{2.5.5}$$

We want to calculate the noise power spectrum

$$S(\varepsilon) = \int_{-\infty}^{\infty} d(t-t') e^{i\varepsilon(t-t')} S(t-t'), \tag{2.5.6}$$

so we start calculating the non-equilibrium noise correlator given by

$$\begin{aligned} S(t,t') &= \langle\{\delta I_L(t), \delta I_L(t')\}\rangle \\ &= \langle\{I_L(t), I_L(t')\}\rangle - 2\langle I_L\rangle^2 \\ &= (ie)^2 \sum_{k,k'} \Big[t_k t'_k \langle \hat{c}_k^\dagger(t)\hat{d}(t)\hat{c}_{k'}^\dagger(t')\hat{d}(t')\rangle - t_k t_{k'}^* \langle \hat{c}_k^\dagger(t)\hat{d}(t)\hat{d}^\dagger(t')\hat{c}_{k'}(t')\rangle \\ &- t_k^* t_{k'} \langle \hat{d}^\dagger(t)\hat{c}_k(t)\hat{c}_{k'}^\dagger(t)\hat{d}(t')\rangle + t_k^* t_{k'}^* \langle \hat{d}^\dagger(t)\hat{c}_k(t)\hat{d}^\dagger(t')\hat{c}_{k'}(t')\rangle + h.c. \Big] - 2\langle I_L\rangle^2, \end{aligned} \tag{2.5.7}$$

and

$$\delta I = I - \langle I\rangle. \tag{2.5.8}$$

From now on we drop the the index L on the operators of the leads and put it back on in the end of the calculation when it is needed. We now define the two particle Green functions

$$G_1(t,t') = -\langle T\hat{c}_k^\dagger(t)\hat{d}(t)\hat{c}_{k'}^\dagger(t')\hat{d}(t')\rangle, \tag{2.5.9}$$

$$G_2(t,t') = -\langle T\hat{c}_k^\dagger(t)\hat{d}(t)\hat{d}^\dagger(t')\hat{c}_{k'}(t')\rangle, \tag{2.5.10}$$

$$G_3(t,t') = -\langle T\hat{d}^\dagger(t)\hat{c}_k(t)\hat{c}_{k'}^\dagger(t')\hat{d}(t')\rangle, \tag{2.5.11}$$

$$G_4(t,t') = -\langle T\hat{d}^\dagger(t)\hat{c}_k(t)\hat{d}^\dagger(t')\hat{c}_{k'}(t')\rangle. \tag{2.5.12}$$

Now we can rewrite the noise in the form

$$\begin{aligned} S(t,t') &= (ie)^2 \sum_{k,k'} [t_k t'_k G_1(t,t') - t_k t_{k'}^* G_2(t,t') \\ &- t_k^* t_{k'} G_3(t,t') + t_k^* t_{k'}^* G_4(t,t')] + h.c. - \langle I_L\rangle^2. \end{aligned} \tag{2.5.13}$$

We start by writing the operators in the interaction picture such that we can perform an S matrix expansion. We perform the calculations on G_2 but the treatment of all other terms

is identical. We start from the definition

$$\begin{aligned} G_2(t,t') &= -\langle T_C \hat{c}_k^\dagger(t)\hat{d}(t)\hat{d}^\dagger(t')\hat{c}_{k'}(t')\rangle \\ &= -\langle T_C \tilde{c}_k^\dagger(t)\tilde{d}(t)\tilde{d}^\dagger(t')\tilde{c}_{k'}(t')S\rangle, \end{aligned} \tag{2.5.14}$$

where $\tilde{c}_k$ indicates that we have written the operator in the interaction picture and T_C is the timer ordering operator that orders over the Keldysh contour in Fig. 2.1. The S-matrix is given by

$$S = \sum_{j=0}^{\infty} \frac{(-i)^j}{j!} \int_C dt_1 \cdots \int dt_j \langle T_C \mathcal{H}_t(t_1)\cdots\mathcal{H}_t(t_2)\rangle, \tag{2.5.15}$$

where $\mathcal{H}_t$ is the tunneling Hamiltonian in eq. (2.5.4). We expand the S-matrix to the second order in the tunneling Hamiltonian and the Green function takes the form

$$\begin{aligned} G_2(t,t') &= -\langle T_C \tilde{c}_k^\dagger(t)\tilde{d}(t)\tilde{d}^\dagger(t')\tilde{c}_{k'}(t')\rangle \\ &+ \frac{1}{2}\langle \tilde{c}_k^\dagger(t)\tilde{d}(t)\tilde{d}^\dagger(t')\tilde{c}_{k'}(t') \\ &\times \int dt_1 \int dt_2 \sum_{k_1 k_2} \Big(t_{k_1}\tilde{c}_{k_1}^\dagger(t_1)\tilde{d}(t_1) + t_{k_1}^*\tilde{d}^\dagger(t_1)\tilde{c}_{k_1}(t_1)\Big) \\ &\times \Big(t_{k_2}\tilde{c}_{k_2}^\dagger(t_2)\tilde{d}(t_2) + t_{k_2}^*\tilde{d}^\dagger(t_2)\tilde{c}_{k_2}(t_2)\Big)\rangle. \end{aligned} \tag{2.5.16}$$

In the interaction picture the operators acting on the level and on the leads are independent and we can therefore split up the expectation value in a resonant level and a lead part. So eq. (2.5.16) can be simplified as

$$\begin{aligned} G_2(t,t') &= -\delta_{kk'} G_k(t',t)\tilde{G}(t,t') \\ &+ \int dt_1 \int dt_2 \sum_{k_1 k_2} \langle T_C \tilde{c}_k^\dagger(t)\tilde{c}_{k'}(t')\tilde{c}_{k_1}^\dagger(t_1)\tilde{c}_{k_2}(t_2)\rangle \\ &\times \langle T_C \tilde{d}^\dagger(t)\tilde{d}(t')\tilde{d}^\dagger(t_1)\tilde{d}(t_2)\rangle. \end{aligned} \tag{2.5.17}$$

We have also introduced the full single particle Green $G_k(t,t') \equiv -i\langle Tc_k^\dagger(t)c_k(t')\rangle$ and the expanded Green function $\tilde{G}_k(t,t') \equiv -i\langle T\tilde{c}_k^\dagger(t)\tilde{c}_k(t')\rangle$. The definitions of the single particle Green functions of the dot are equivalent. Up to this point the noise calculation is the same for both non-interacting systems and interacting systems. The change comes when start to try to split up the two particle Green functions into single particle Green functions. In this section we deal with non-interacting electrons: so we can just split them up using Wick's theorem without complication. For the interacting problem things are more complicated but we get to that in chapter 4. In both cases we have non-interacting leads so we start by factorising the two particle Green function for the leads:

$$\begin{aligned}
\langle T_C\tilde{c}_k^\dagger(t)\tilde{c}_{k'}(t')\tilde{c}_{k_1}^\dagger(t_1)\tilde{c}_{k_2}(t_2)\rangle &= \langle T_C\tilde{c}_k^\dagger(t)\tilde{c}_{k'}(t')\rangle\langle T_C\tilde{c}_{k_1}^\dagger(t_1)\tilde{c}_{k_2}(t_2)\rangle \\
&- \langle T_C\tilde{c}_k^\dagger(t)\tilde{c}_{k_2}(t_2)\rangle\langle T_C\tilde{c}_{k_1}^\dagger(t_1)\tilde{c}_{k'}(t')\rangle \\
&= -\delta_{kk'}\delta_{k_1k_2}\tilde{G}_k(t',t)\tilde{G}_{k_1}(t_2,t_2) \qquad (2.5.18)\\
&+ \delta_{kk_2}\delta_{k'k_1}\tilde{G}_k(t_2,t)\tilde{G}_{k'}(t',t_1).
\end{aligned}$$

We can now combine the first term of eq. (2.5.18) and combine it with eq. (2.5.17) to obtain

$$-\delta_{kk'}G_k(t',t)\left[\tilde{G}(t,t') - \sum_{k_1}|t_{k_1}|^2\int dt_1\int dt_2 G_{k_1}(t_2,t_1)\langle T_C\tilde{d}_k^\dagger(t)\tilde{d}_{k'}(t')\tilde{d}_{k_1}^\dagger(t_1)\tilde{d}_{k_2}(t_2)\rangle\right].$$

The terms in the square bracket we recognise as the first two terms of a series expansion of the full Green function of an interacting central region. If one carefully does the combinatorics of the terms of the full expansion of the tunneling Hamiltonian one obtains the full series expansion $G(t,t')$. So these terms sum up to $G_k('t,t)G(t,t')$. The second term in eq. (2.5.18) gives the zeroth order term of the expansion of the full two particle Green function and the higher order terms of the S-matrix expansion will give the full two particle Green function.

Putting all this together we now write eq. (2.5.17) in the form:

$$\begin{aligned} G_2(t,t') &= -\delta_{kk'} G_k(t',t) G(t,t') \\ &- \int dt_1 \int dt_2 t_{k'} t_k^* G_k(t_2,t) G_{k'}(t,t') G_2^{dd}(t,t',t_1,t_2), \end{aligned} \tag{2.5.19}$$

where

$$G_2^{dd}(t,t',t_1,t_2) = \langle T_C \hat{d}(t) \hat{d}^\dagger(t') \hat{d}(t_1) \hat{d}^\dagger(t_2) \rangle. \tag{2.5.20}$$

Performing the same analysis for the other terms in eq. (2.5.13) and introducing the two particle Green functions:

$$G_1^{dd}(t,t',t_1,t_2) = \langle T_C \hat{d}(t) \hat{d}(t') \hat{d}^\dagger(t_1) \hat{d}^\dagger(t_2) \rangle, \tag{2.5.21}$$

$$G_3^{dd}(t,t',t_1,t_2) = \langle T_C \hat{d}^\dagger(t) \hat{d}(t') \hat{d}(t_1) \hat{d}^\dagger(t_2) \rangle, \tag{2.5.22}$$

$$G_4^{dd}(t,t',t_1,t_2) = \langle T_C \hat{d}^\dagger(t) \hat{d}^\dagger(t') \hat{d}(t_1) \hat{d}(t_2) \rangle, \tag{2.5.23}$$

we can now write the noise in the form:

$$\begin{aligned} S(t,t') &= 4 \Bigg\{ \sum_k |t_k|^2 \left[G_k(t',t) G_n(t,t') + G_k(t,t') G_n(t',t) \right] \\ &+ \sum_{k,k'} |t_k|^2 |t_{k'}|^2 \int dt_1 \int dt_2 \\ &\times \big[-G_k(t_1,t) G_k(t_2,t') G_1^{dd}(t,t',t_1,t_2) \\ &+ G_k(t_2,t) G_k(t',t_1) G_2^{dd}(t,t',t_1,t_2) \\ &- G_k(t,t_1) G_k(t_2,t') G_1^{dd}(t,t',t_1,t_2) \\ &- G_k(t,t_1) G_k(t',t_2) G_1^{dd}(t,t',t_1,t_2) \big] \Bigg\} + h.c. - 2\langle I_L \rangle^2. \end{aligned} \tag{2.5.24}$$

where we have introduced the index n on the single particle Green functions of the dot to make them easier to separate from the lead Green functions. Using Wick's theorem we can

now split the two particle Green functions into single particle Green functions

$$G_1^{dd}(t,t',t_1,t_2) = G_n(t,t_2)G_n(t',t_1) - G_n(t,t_1)G_n(t',t_2), \tag{2.5.25}$$

$$G_2^{dd}(t,t',t_1,t_2) = G_n(t,t')G_n(t_1,t_2) - G_n(t,t_2)G_n(t_1,t'), \tag{2.5.26}$$

$$G_3^{dd}(t,t',t_1,t_2) = G_n(t_1,t)G_n(t',t_2) - G_n(t',t)G_n(t_1,t_2), \tag{2.5.27}$$

$$G_4^{dd}(t,t',t_1,t_2) = G_n(t_2,t)G_n(t_1,t') - G_n(t_1,t)G_n(t_2,t'). \tag{2.5.28}$$

Inserting eqs. (2.5.25)-(2.5.28) into eq. (2.5.24) we obtain a large group of unconnected terms and a large group of connected terms. The treatment of both types of terms are similar with the difference that the unconnected terms are cancelled exactly with the $-2\langle I_L\rangle^2$ and the connected terms give interesting contributions to the noise.

2.5.1 Unconnected terms

We start with analysing the unconnected terms

$$\begin{aligned} S_{un}(t,t') &= e^2 \left\{ \sum_{k,k'} |t_k|^2 |t_{k'}|^2 \int dt_1 \int dt_2 \right. \\ &\quad \times [G_k(t_1,t)G_k(t_2,t')G_n(t,t_1)G_n(t',t_2) \\ &\quad -G_k(t_2,t)G_k(t',t_1)G_n(t,t_2)G_n(t_1,t') \\ &\quad -G_k(t,t_1)G_k(t_2,t')G_n(t_1,t)G_n(t',t_2) \\ &\quad \left. +G_k(t,t_1)G_k(t',t_2)G_n(t_1,t)G_n(t_2,t')] \right\}, \end{aligned} \tag{2.5.29}$$

where the standard analytic continuation rules are ill-defined since we only have one time label. This can be solved by carefully taking a look at the origin of each term. Take the term

$$\int dt_1 G_k(t_1,t)G_n(t,t_1), \tag{2.5.30}$$

as an example, we see that t in G_k has to be greater than t in G since $c_k^\dagger(t)$ should be left of $d(t)$. We can now rewrite eq. (2.5.30) as

$$\int dt_1 G_k(t_1, t^+) G(t, t_1). \tag{2.5.31}$$

Rearranging the terms we once again have a form where we can use the analytic continuation rules

$$\begin{aligned} \left[\int dt_1 G_n(t, t_1) G_k(t_1, t^+)\right]^< &= \int dt_1 \left(G^R(t, t_1) G_k^<(t_1, t) + G^<(t, t_1) G_k^A(t_1, t)\right) \\ &\equiv G_{n,k}^<(t, t'). \end{aligned} \tag{2.5.32}$$

Performing this analysis on all the terms in the eq. (2.5.30) we obtain the final results for the unconnected terms

$$\begin{aligned} S_{un}(t, t') &= e^2 \sum_{k,k'} |t_k|^2 |t_k'|^2 \left\{G_{nk}^<(t, t) G_{nk'}^<(t', t') - G_{nk}^<(t, t) G_{k'n}^<(t', t')\right. \\ &- \left. G_{kn}^<(t, t) G_{nk'}^<(t', t') + G_{kn}^<(t, t) G_{k'n}^<(t', t')\right\} \\ &= 2e^2 \sum_{k,k'} |t_k|^2 |t_k'|^2 \left\{\left[G_{nk}^<(t, t) - G_{kn}^<(t, t)\right] \left[G_{nk'}^<(t', t') - G_{k'n}^<(t', t')\right]\right\} \\ &\equiv 2\langle I_L \rangle^2. \end{aligned} \tag{2.5.33}$$

Wee see that the unconnected terms exactly cancel with the $2\langle I_L \rangle^2$ term in the definition of the noise. This is an important result since without this cancelation the unconnected terms would cause a anomalous zero frequency delta peak. This is equivalent to the cancellation of the diamagnetic terms in the Kubo formula by the unconnected current-current correlators.

2.5.2 Connected terms

What now remains are the connected terms which are the terms that will contribute to the noise:

$$\begin{aligned} S(t,t') &= e^2 \left[\sum_k |t_k|^2 \left(G_k(t,t')G_n(t',t) + G_n(t,t')G_k(t',t) \right) \right. \\ &+ \sum_{k,k'} |t_k|^2 |t_{k'}|^2 \int dt_1 dt_2 \\ &\times \{ -G_k(t_1,t)G_{k'}(t_2,t')G_n(t,t_2)G_n(t',t_1) \\ &+ G_k(t_2,t)G_{k'}(t',t_1)G_n(t,t')G_n(t_1,t_2) \\ &+ G_k(t,t_1)G_{k'}(t_2,t')G_n(t',t)G_n(t_1,t_2) \\ &- \left. G_k(t,t_1)G_{k'}(t',t_2)G_n(t_2,t)G_n(t_1,t') \} + h.c. \right] \end{aligned} \tag{2.5.34}$$

Fixing $t > t'$ we can use the analytic continuation rules derived in the previous section. We start with the trivial case

$$\left[G_k(t,t')G_n(t',t) + G_n(t,t')G_k(t',t) \right]_{t>t'} = G_k^<(t',t)G_n^>(t,t') + G_k^>(t,t')G_n^<(t',t). \tag{2.5.35}$$

In the case of terms that have products of 4 Green functions we have two type terms. The first type is of the form

$$\left[\int dt_1 G_n(t',t_1)G_k(t_1,t) \int dt_2 G_n(t,t_2)G_k(t_2,t') \right]_{t>t'}. \tag{2.5.36}$$

In this case we can treat the product as two separate integrals and use the relationship eq. (2.4.11) to obtain

$$\begin{aligned} &\left[\int dt_1 G_n(t',t_1)G_k(t_1,t) \int dt_2 G_n(t,t_2)G_k(t_2,t') \right]_{t>t'} \\ &= \int dt_1 \left[G_n^R(t',t_1)G_k^R(t_1,t) + G_n^<(t',t_1)G_k^A(t_1,t) \right] \end{aligned} \tag{2.5.37}$$

$$\times \int dt_2 \left[G_n^R(t,t_2) G_k^>(t_2,t') + G_n^>(t,t_2) G_k^A(t_2,t') \right]. \tag{2.5.38}$$

The second type is of the form

$$\left[G_n(t,t') \int dt_1 \int dt_2 G_k(t',t_1) G_n(t_1,t_2) G_k(t_2,t) \right]_{t>t'}, \tag{2.5.39}$$

which is of the structure $D = ABC$, so we insert eq. (2.4.13) into eq. (2.5.39) and obtain

$$\begin{aligned}
&\left[G_n(t,t') \int dt_1 \int dt_2 G_k(t',t_1) G_n(t_1,t_2) G_k(t_2,t) \right]_{t>t'} \\
&= G^>(t,t') \int dt_1 \int dt_2 \left[G_k^R(t',t_1) G_n^R(t_1,t_2) G_k^<(t_2,t) \right. \\
&\qquad + G_k^R(t',t_1) G_n^<(t_1,t_2) G_k^A(t_2,t) \\
&\qquad \left. + G_k^<(t',t_1) G_n^A(t_1,t_2) G_k^A(t_2,t). \right.
\end{aligned} \tag{2.5.40}$$

Applying the analytic continuation techniques on eq. (2.5.34) and taking the Fourier transform and then the zero frequency limit we obtain

$$\begin{aligned}
S(0) &= 2e^2 \int \frac{d\varepsilon}{2\pi} \left\{ i f_L \Gamma_L \left[G^R(\varepsilon) - G^A(\varepsilon) \right] + i \left[2 f_L(\varepsilon) - 1 \right] G^<(\varepsilon) \right. \\
&+ \left[G^R(\varepsilon) - G^A(\varepsilon) \right] f_L(\varepsilon) \Gamma_L \left[G^R(\varepsilon) - G^A(\varepsilon) \right] \Gamma_L \\
&+ \left[G^R(\varepsilon) - G^A(\varepsilon) \right] (2 f_L(\varepsilon) - 1) \Gamma_L G^<(\varepsilon) \Gamma_L \\
&- f_L(\varepsilon)[1 - f_L(\varepsilon)] \left[G^A(\varepsilon) \Gamma_L G^A(\varepsilon) \Gamma_L + G^R(\varepsilon) \Gamma_L G^R(\varepsilon) \Gamma_L \right] \\
&+ G^<(\varepsilon) \Gamma_L \left[G^R(\varepsilon) - G^A(\varepsilon) \right] \Gamma_L \\
&+ \left. G^<(\varepsilon) \Gamma_L G^<(\varepsilon) \Gamma_L \right\},
\end{aligned} \tag{2.5.41}$$

where we have introduced

$$\Gamma_L = 2\pi \sum_k |t_k|^2 \delta(\varepsilon - \varepsilon_k), \tag{2.5.42}$$

and inserted the non-interacting lead Green functions [38]

$$\Sigma^{R,A}_{\alpha,m,n}(\varepsilon) \equiv \sum_k |t_k|^2 G^{R,A}_k = \Delta^{\alpha}_{m,n} \mp \frac{i}{2}\Gamma_{Lmn}, \tag{2.5.43}$$

$$\Sigma^{<}_{\alpha,m,n}(\varepsilon) \equiv \sum_k |t_k|^2 G^{<}_k = i\Gamma_{Lmn} f_L(\varepsilon), \tag{2.5.44}$$

$$\Sigma^{>}_{\alpha,m,n}(\varepsilon) \equiv \sum_k |t_k|^2 G^{>}_k = -i\Gamma_{Lmn}(1 - f_L(\varepsilon)). \tag{2.5.45}$$

Introducing the identities

$$G^{<}(\varepsilon) = iG^R \left[f_L(\varepsilon)\Gamma_L + f_R\Gamma_R\right] G^A(\varepsilon), \tag{2.5.46}$$

$$G^R(\varepsilon) - G^A(\varepsilon) = -iG^R(\varepsilon)[\Gamma_L + \Gamma_R]G^A(\varepsilon), \tag{2.5.47}$$

$$G^A(\varepsilon)\Gamma_L G^A(\varepsilon)\Gamma_L + G^R(\varepsilon)\Gamma_L G^R(\varepsilon)\Gamma_L \tag{2.5.48}$$

$$= [G^R(\varepsilon) - G^A(\varepsilon)]\Gamma_L[G^R(\varepsilon) - G^A(\varepsilon)]\Gamma_L + 2G^R(\varepsilon)\Gamma_L G^A(\varepsilon)\Gamma_L, \tag{2.5.49}$$

$$T(\varepsilon) = \Gamma_L G^A(\varepsilon)\Gamma_R G^R(\varepsilon). \tag{2.5.50}$$

where $T(\varepsilon)$ is the transmission probability. The results can finally be written in the standard form [8]

$$\begin{aligned} S(0) &= 2e^2 \int \frac{d\varepsilon}{2\pi} \left\{ \left\{ f_L(\varepsilon)(1 - f_L(\varepsilon)) + f_R(\varepsilon)(1 - f_R(\varepsilon)) \right\} T(\varepsilon) \right. \\ &+ \left. \left(f_L(\varepsilon) - f_R(\varepsilon)\right)^2 T(\varepsilon)(1 - T(\varepsilon)) \right\}. \end{aligned} \tag{2.5.51}$$

The first term is the thermal noise since it vanishes at zero temperature, and the second part is the non-equilibrium contribution that vanishes at zero bias.

2.6 Summary

In this chapter we have introduced the basic theory of Green functions. The focus was on non-equlibrium Green functions and we derived the functional integral form of Keldysh

Green functions. We also introduced analytic continuation rules and closed the chapter by calculating the noise in a resonant level model. The noise consists of two parts: one that comes from thermal fluctuations and one that comes from non-equilibrium effects. The second type of noise is the important type and will give us important information about the interactions in the chapters to come. The main formula of this section, eq. (2.5.51), will be very useful in chapter 4 when we study the noise of a Coulomb blockaded quantum dot.

Chapter 3

TUNNELING DENSITY OF STATES

The goal of this chapter is to calculate the tunneling density of states (TDoS) near the degeneracy point in a Coulomb blockaded quantum dot using two different approaches. The first method, developed by Sedlmayr et al. [1], is defined using a functional integral approach in the Keldysh technique. This reproduction of the key results from Sedlmayr *et al.*[1] will then be used as a reference point when we develop a new method to calculate the Green functions and TDoS for this system. This new technique will be crucial in the next chapter, for our ability to find the pair correlations functions that appear in the noise of a Coulomb blockaded quantum dot.

3.1 The Sedlmayr-Yurkevich-Lerner method

Keeping only the charging term in the universal Hamiltonian [14] of a zero dimensional system our Hamiltonian takes the form

$$\mathcal{H} = \mathcal{H}_0 + \frac{E_c}{2}\left(\hat{N} - N_g\right)^2. \tag{3.1.1}$$

Where

$$\mathcal{H}_0 = \sum_n \varepsilon_n \psi_n^\dagger \psi_n, \tag{3.1.2}$$

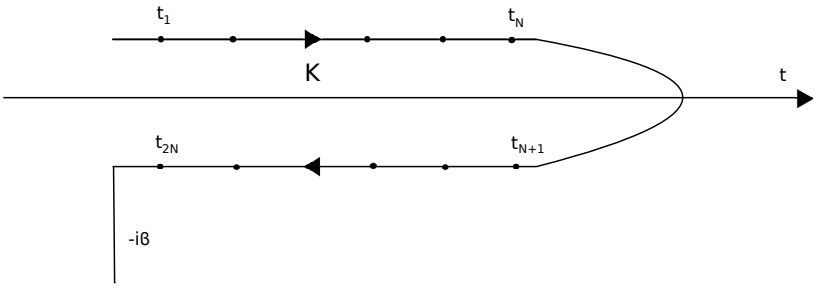


Figure 3.1: The interaction Keldysh contour.

is the Hamiltonian of free electrons in a random potential, due to electron motion, V, $\hat{N}$ is the number operator and eN_g is the neutralising background charge (governed by the gate voltage, V_g). This system can also be described in terms of its action $S[\psi] = S_0[\psi] + S_c[\psi]$ given by

$$S_0[\psi] = \int_K dt \int dr \bar{\psi}(r,t) \left[i\partial_t - \hat{\xi} \right] \psi(r',t'), \qquad \hat{\xi} \equiv \frac{p^2}{2m} + V - \tilde{\mu}, \tag{3.1.3}$$

$$S_c[\psi] = -\frac{E_c}{2} \int_K dt N^2(t), \qquad N(t) = \int dr \bar{\psi}(r,t)\psi(r,t). \tag{3.1.4}$$

where K is the interaction Keldysh contour [37], see Fig. 3.1. To calculate the observables of the system we start from the definition of the Green functions in the functional integral formulation in the Keldysh technique

$$iG(r,t;r',t') = Z^{-1} \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\psi(r,t)\bar{\psi}(r',t')e^{iS[\psi]}, \tag{3.1.5}$$

where

$$Z = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{iS[\psi]}, \tag{3.1.6}$$

is the partition function. In the zero-dimensional regime we can expand the Grassmann fields in terms of free electron eigenfunctions $c_n(t)$ and $\bar{c}_n(t)$ that depend on time only,

$$\psi(r,t) = \sum_n \psi_n(r) c_n(t), \quad \hat{\xi}\psi_n(r) = \xi_n \psi_n(r), \quad \xi_n = \varepsilon_n - \tilde{\mu}. \tag{3.1.7}$$

We can now rewrite the Green function as

$$G(r,t;r',t') = \sum_n \psi_n(r)\bar{\psi}_n(r') G_n(t,t'). \tag{3.1.8}$$

Since we are considering the quantum dot as a zero dimensional object the two positions r and r' are indistinguishable so the observable quantities can be found from

$$G(t,t') \equiv \int d^d r G(r,t;r',t') = \sum_n G_n(t,t'). \tag{3.1.9}$$

To be able to calculate the Green functions we have to decouple the charging term in the action and make it quadratic. This is done by the Hubbard-Stratonovich transformation by introducing the identity

$$1 = \frac{E_c}{2\pi} \int d\phi \exp\left(-\frac{i}{2}\phi \frac{1}{E_c}\phi\right), \tag{3.1.10}$$

and we make a shift in the bosonic field $\phi \to \phi + iE_cN$ to rewrite Eq.(3.1.10) as

$$1 = \frac{E_c}{2\pi} \int d\phi \exp\left(-\frac{i}{2E_c}\phi^2 - i^2\phi N + \frac{E_c}{2}N^2\right). \tag{3.1.11}$$

Multiplying the Green function by eq. (3.1.11) we obtain a quadratic form of the Green function

$$iG_n(t,t') = \frac{\int \mathcal{D}\phi e^{iS_c[\phi]} \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \psi_n(t)\psi_n(t') e^{iS[\bar{\psi},\psi,\phi]}}{\int \mathcal{D}\phi e^{iS_c[\phi]} \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{iS[\bar{\psi},\psi,\phi]}}, \tag{3.1.12}$$

where we have split the action into two parts. One that contains both a fermionic and bosonic part

$$S_0[\phi, \bar{c}_n, c_n] = \int_K dt \bar{c}_n(t) \left[i\partial_t - i\phi - \xi_n \right] c_n(t), \tag{3.1.13}$$

and one contains a purely bosonic part:

$$S_c[\phi] = -\frac{1}{2E_c} \int_K dt \phi^2(t), \tag{3.1.14}$$

where K is the interacting Keldysh contour, see Fig. 3.1. We now have an action that is quadratic in the fermionic fields so we can integrate them out and leaving only the bosonic part to be solved. In the previous chapter, see eq. (2.4.6), we proved the following identities

$$Z_0^{-1} \int \mathcal{D}\bar{c}_n \mathcal{D}c_n e^{i\int_K dt \bar{c}_n [i\delta_t + \varphi] c_n(t)} = 1 + e^{i\int_K dt \varphi(t)}, \tag{3.1.15}$$

$$Z_0^{-1} \int \mathcal{D}\bar{c}_n \mathcal{D}c_n c_n(t) \bar{c}_n(t') e^{i\int_K dt \bar{c}_n [i\delta_t + \varphi] c_n(t)} = \mathrm{sgn}(t, t') e^{i\int_{t'}^{t} d\tau \varphi(\tau)}. \tag{3.1.16}$$

Using these identities we see that after doing the fermionic field integrals we can write eq. (3.1.12) in the form

$$\begin{aligned} G_n(t, t') &= -\frac{i\mathrm{sgn}(t, t')}{Z} \int \mathcal{D}\phi e^{i\int_{t'}^{t} d\tau (-\xi_n - i\phi(\tau))} e^{iS[\phi]} \prod_{m \neq n} \left(1 + e^{i\int_K dt (-i\phi(t) - \xi_n)} \right) \\ &= -\frac{i\mathrm{sgn}(t, t')}{Z} \int \mathcal{D}\phi e^{\int_{t'}^{t} (-i\xi_n + \phi)} \Xi_n(\phi_0) e^{iS[\phi]}, \end{aligned} \tag{3.1.17}$$

where

$$Z = \int \mathcal{D}\phi e^{iS[\phi]} \Xi(\phi_0). \tag{3.1.18}$$

Here we have defined the grand-canonical partition function, $\Xi(\phi_0)$, and the grand canonical partition function with the n-th level excluded, $\Xi_n(\phi_0)$, with energy levels in both shifted

by the charging effects:

$$\Xi(\phi_0) \equiv \prod_m \left(1 + e^{-\beta\xi_m + \phi_0}\right) \qquad \Xi_n(\phi_0) \equiv \Xi(\phi_0)\left(1 + e^{-\beta\xi_n + \phi_0}\right)^{-1}, \tag{3.1.19}$$

where

$$\phi_0 = \int_K dt\phi(t). \tag{3.1.20}$$

It is now convenient to expand the grand canonical partition functions in terms of the canonical partition functions

$$\Xi(\phi_0) = \sum_{N=0}^{\infty} Z_N e^{(\beta\tilde{\mu}+\phi_0)N}, \qquad Z_N = \oint \frac{d\theta}{2\pi} e^{-iN\theta} \prod_m \left(1 + e^{-\beta\varepsilon + i\theta}\right), \tag{3.1.21}$$

$$\Xi_n(\phi_0) = \sum_{N=0}^{\infty} Z_N(\varepsilon_n) e^{(\beta\tilde{\mu}+\phi_0)N}, \qquad Z_N(\varepsilon_n) = \oint \frac{d\theta}{2\pi} e^{-iN\theta} \prod_{m\neq n} \left(1 + e^{-\beta\varepsilon + i\theta}\right). \tag{3.1.22}$$

Inserting the canonical partition functions into the Green functions in eq. (3.1.17) we obtain

$$\begin{aligned} G_n(t,t') &= -\frac{i\mathrm{sgn}(t,t')}{Z} \sum_{N=0}^{\infty} \int \mathcal{D}\phi e^{\int_{t'}^{t} d\tau(-i\xi+\phi(\tau))} \\ &\quad \oint \frac{d\theta}{2\pi} e^{-iN\theta} \prod_{m\neq n} \left(1 + e^{-\beta\varepsilon_m + i\phi_0}\right) e^{(\beta\tilde{\mu}+\phi_0)N} e^{iS[\phi_0]} \\ &= -\frac{i\mathrm{sgn}(t,t')}{Z} \sum_{N=0}^{\infty} Z_N(\varepsilon_n) e^{\beta\tilde{\mu}N} \\ &\quad \int \mathcal{D}\phi e^{\int_{t'}^{t} d\tau(-i\xi+\phi(\tau))} e^{i\int_K d\tau\left(-i\phi(\tau)N - \frac{1}{2E_c}\phi^2(\tau)\right)}. \end{aligned} \tag{3.1.23}$$

We now just have to make the bosonic field integral Gaussian. This is straightforward to do if we define the order of the time components t and t'. For the TDoS we are interested in

$G^>$ and $G^<$ so we define

$$G(t,t') \equiv \sum_n G_n(t,t') \equiv \begin{cases} G^<(t,t') & t' > t \\ G^>(t,t') & t > t' \end{cases}, \tag{3.1.24}$$

the time ordered Green function. Let's start with the $G^>$ and we complete the square

$$\begin{aligned} G_n^>(t,t') &= -\frac{i}{Z}\sum_{N=0}^{\infty} Z_N(\varepsilon_n) e^{\beta\tilde{\mu}N} \\ &\quad \times \int \mathcal{D}\phi e^{i\int_K d\tau\left(-i\phi(\tau)N - i(\theta(\tau-t')-\theta(\tau-t))(-i\xi+\phi(\tau)) - \frac{1}{2E_c}\phi^2(\tau)\right)} \\ &= -\frac{i}{Z}\sum_{N=0}^{\infty} Z_N(\varepsilon_n) e^{\beta N\left(\tilde{\mu}-\frac{1}{2}NE_c\right)} e^{-i(t-t')\left(\xi+E_cN+\frac{E_c}{2}\right)} \\ &\quad \times \int \mathcal{D}\phi e^{-\frac{i}{2E_c}\int_K d\tau\{\phi(\tau)+iE_c(N+(\theta(\tau-t')-\theta(\tau-t))\}^2}. \end{aligned} \tag{3.1.25}$$

Then we can now easily do the ϕ integral since it is Gaussian and we then take the Fourier transform and obtain

$$\begin{aligned} G_n^>(\varepsilon) &= -\frac{i}{Z}\sum_{N=0}^{\infty} Z_N(\varepsilon_n) e^{\beta N(\tilde{\mu}-\frac{1}{2}NE_c)} \int dt e^{it\left(\varepsilon-\xi-E_cN-\frac{E_c}{2}\right)} \\ &= -\frac{i}{Z}\sum_{N=0}^{\infty} Z_N(\varepsilon_n) e^{\beta N\left(\tilde{\mu}-\frac{1}{2}NE_c\right)} \delta\left(\varepsilon-\xi-E_cN-\frac{E_c}{2}\right) \\ &= -\frac{i}{Z}\sum_{N=0}^{\infty} Z_N(\varepsilon-\Omega_N) e^{-\beta E_N} \delta\left(\varepsilon-\varepsilon_n-\Omega_N\right), \end{aligned} \tag{3.1.26}$$

where

$$E_N \equiv \frac{E_c}{2}(N-N_g)^2 - \mu, N \qquad \Omega_N \equiv E_c\left(N+\frac{1}{2}-N_g\right). \tag{3.1.27}$$

The calculation for $G^<$ is identical but we will instead use a trick to get it from the greater then Green function instead. In equilibrium we know that $G_n^<(\varepsilon) = -e^{-\beta(\varepsilon-\mu)}G_n^>(\varepsilon)$. But in the Coulomb blockade regime both of these functions are very sharp so it is not a very good

relationship to use. Instead we use that $Z_N = Z_N(\varepsilon_n) + e^{-\beta\varepsilon_n} Z_{N-1}(\varepsilon_n)$ which follows from eqs. (3.1.21) and (3.1.22). After a straightforward transformation we arrive at the result

$$G_n^<(\varepsilon) \;=\; -\frac{i}{Z}\sum_{N=0}^{\infty} Z_N(\varepsilon - \Omega_{N-1}) e^{-\beta E_N - \beta(\varepsilon-\mu)} \delta\left(\varepsilon - \varepsilon_n - \Omega_{N-1}\right). \tag{3.1.28}$$

The formal definition of $Z_N(\varepsilon_n)$ is

$$\frac{Z_N(\varepsilon_n)}{Z_N} = \frac{Tr_N\left(c_n c_n^\dagger e^{-\beta H_0}\right)}{Tr_N e^{-\beta H_0}} = 1 - F_N(\varepsilon_n), \tag{3.1.29}$$

where $F_N(\varepsilon_n)$ is the distribution function of a system of N non-interacting electrons. Now we can average over disorder simply by replacing the mean TDoS of non-interacting electrons ν_0, with $\sum_n \delta(\varepsilon - \varepsilon_n - \Omega_N)$, with the assumption that the TDoS is smooth in any realisation of disorder, which is valid when the mean level spacing is much smaller then T. We now obtain

$$G^>(\varepsilon) = \frac{2\pi\nu_0}{Z}\sum_N e^{-\beta E_N}\left(1 - F_N(\varepsilon - \Omega_N)\right). \tag{3.1.30}$$

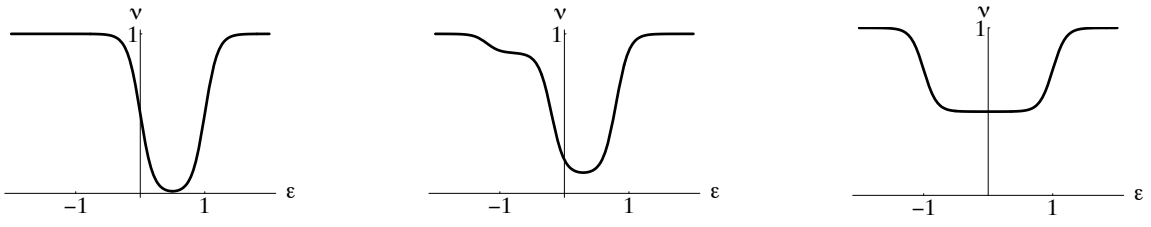
Since the number of electrons is large, $N \ll 1$, the distribution function is approximatively the same as the Fermi-Dirac distribution function $f(\varepsilon - \Omega_N)$ with the chemical potential of order $N\delta$, which is negligible compared to Ω_N. This gives the final form of the Green functions of the dot,

$$G^>(\varepsilon) = -\frac{2\pi\nu_0}{Z}\sum_N e^{-\beta E_N}(1 - f(\varepsilon - \Omega_N)), \tag{3.1.31}$$

$$G^<(\varepsilon) = \frac{2\pi\nu_0}{Z}\sum_N e^{-\beta E_N}(f(\varepsilon - \Omega_{N-1})). \tag{3.1.32}$$

The TDoS can now be obtained from the standard relationship

$$\nu(\varepsilon) = \frac{i}{2\pi}\left(G^R(\varepsilon) - G^A(\varepsilon)\right) = \frac{i}{2\pi}\left(G^>(\varepsilon) - G^<(\varepsilon)\right). \tag{3.1.33}$$



(a) (b) (c)

Figure 3.2: The dependence of the TDoS on the energy, measured in E_c, is plotted in three different regions: (a) in the valley, (b) through an intermediate region, and (c) at the peak. Taken from [1]

Substituting the Green functions eqs. (3.1.31) and (3.1.32) into eq. (3.1.33) we obtain

$$\frac{\nu(\varepsilon)}{\nu_0} = \frac{1}{Z}\sum_N e^{-\beta E_N}\left(f(\varepsilon - E_c(N - 1/2 - N_g)) + 1 - f(\varepsilon - E_c(N + 1/2 - N_g))\right) \quad (3.1.34)$$

We keep the leading order terms in the sum over N since all other terms will be exponentially suppressed,

$$\frac{\nu(\varepsilon)}{\nu_0} = \frac{U(\varepsilon - \Omega_N) + e^{-\beta\delta_\mu}U(\varepsilon - \Omega_{N+1})}{1 + e^{-\beta\delta_\mu}}, \quad (3.1.35)$$

where we have defined $U(\varepsilon - \Omega_N) = f(\varepsilon - \Omega_{N-1}) + 1 - f(\varepsilon - \Omega_N)$. In figure Fig. 3.2(a) we are away from the degeneracy point and we are down in the conductance valley. At this point the $e^{-\beta\delta_\mu}$ term is suppressed so we only get contributions from one of the terms in eq. (3.1.35), this give us the gap that we see in Fig. 3.2(a). This gap is them smeared when we approach the degeneracy point when we get contributions from both the terms in eq. (3.1.35), as can be seen in Fig. 3.2(b). At the peak of conductance the TDoS is finite for all energies but shows a half gap at $|\varepsilon| < E_c$.

3.2 The new approach

For the TDoS this approach works very well, but for the noise the bosonic field introduced in the Hubbard-Stratonovich transformation cause problems. In the next chapter we will discuss how this happens in detail. But in this section we will introduce a new method to calculate the Green functions without doing a Hubbard-Stratonovich transformation. This can be used to circumvent the problems caused by it when we calculate the noise in the next chapter.

The starting point is to rewrite the universal Hamiltonian [14] by connect the tunneling to a single level of the quantum dot and separate this level from the rest of the Hamiltonian in the following way:

$$\begin{aligned}
\mathcal{H}_0 &= \sum_{m\neq n} \xi_m d_m(t) d_m(t), && (3.2.1)\\
\mathcal{H}(E_N) &= \frac{E_c}{2}\left(N - N_g\right)^2, && (3.2.2)\\
\mathcal{H}_n &= \xi_n d_n^\dagger d_n(t) + \Omega(N') d_n(t)^\dagger d_n(t) + \sum_{\alpha,k} t_{\alpha,n} d_n(t)^\dagger c_{\alpha k}(t) + h.c. && (3.2.3)\\
\mathcal{H}_0^{lead} &= \sum_{\alpha k} \varepsilon_k c_{\alpha k}^\dagger c_{\alpha k} && (3.2.4)
\end{aligned}$$

The point of the Hamiltonian in this form is that we get two parts: one that depends on the number of particles and one that depends on the distribution of the quantum dot. To illustrate how this redefinition can be used we start from the definition of the Green function of the dot:

$$G_n(t,t') = \frac{Tr\left(e^{-\beta H} d_n(t) d_n^\dagger(t')\right)}{Tr\left(e^{-\beta H}\right)}. \qquad (3.2.5)$$

Since the term $\mathcal{H}(E_N)$ is the only term that depends on the number of electrons on the dot it will commute with the rest of the Hamiltonian and we can use the properties of the trace,

i.e. $Tr(X \otimes Y) = Tr(X)Tr(Y)$, and pull the interaction term out front. Using the result that

$$Tr(e^{-\beta\mathcal{H}(E_N)}) = \sum_N e^{-\beta E_N}, \tag{3.2.6}$$

we can write the Green function of the dot as

$$G_n(t,t') = \frac{1}{Z}\sum_N e^{-\beta E_N} Tr_N\left(e^{-\beta\mathcal{H}_0}\right) Tr_n\left(e^{-\beta(\mathcal{H}_0^{lead}+\mathcal{H}_n)} d_n(t) d_n^\dagger(t')\right). \tag{3.2.7}$$

The first trace is a sum over all the levels except level n and this is equivalent to the canonical partition function with level n removed, i.e.,

$$Z_N(\varepsilon_n) = Tr_N\left(e^{-\beta\mathcal{H}_0}\right). \tag{3.2.8}$$

In the second trace we can multiply and divide by the canonical partition function of a resonant level model to obtain

$$\frac{Tr_n\left(e^{-\beta(\mathcal{H}_0^{lead}+\mathcal{H}_n)} d_n(t) d_n^\dagger(t')\right)}{Tr\left(e^{-\beta(\mathcal{H}_0^{lead}+\mathcal{H}_n)}\right)} Z_{RL} = G_{RL}(t,t') Z_{RL}. \tag{3.2.9}$$

where Z_{RL} is the partition function of the resonant level model and G_{RL} is the Green function of the resonant level discussed in chapter 2. At this stage we see that we have succeed in rewriting the problem as a resonant level model and the interaction plays the role of weights of the different charge states. We can now rewrite eq. (3.2.7) in the form

$$G_{nn}(t,t') = \frac{1}{Z}\sum_N e^{-\beta E_N} Z_N(\varepsilon_n) G_{RL}(t,t') Z_{RL}, \tag{3.2.10}$$

where the partition function now can be written as

$$\mathcal{Z} = \sum_N e^{-\beta E_N} Z_N(\varepsilon) Z_{RL}. \tag{3.2.11}$$

Using this form of the Green functions, we can now calculate the TDoS and compare this to the results of Sedlmayr *et al.* [1]. The starting point is to neglect the off diagonal terms of the Green function i.e. we define the retarded Green function as

$$G^R(\varepsilon) = \sum_n G^R_{nn}(\varepsilon). \tag{3.2.12}$$

This assumption is reasonable for a non-interacting system where this is simply equivalent to resonant tunneling [13]. When we have a weak coupling to the dot resonant tunneling is the regime where the average distance between the resonances, Δ, is much greater than the width of the resonance. Only the level that is closest to the scattering energy is contributing to the transport. The partition function of the resonant level model can be written

$$\mathcal{Z} = 1 + e^{-\beta(\xi_n + \Omega_N)}. \tag{3.2.13}$$

We now split the Green function into the two situations where the nth level is either empty or occupied.

$$G^R(\varepsilon) = \sum_n \frac{1}{\mathcal{Z}} \left\{ \sum_N Z_N(\varepsilon_n) \left(e^{-\beta E_N} G^R_{RL}(\varepsilon, \Omega_N) + e^{-\beta(E_{N+1} + \varepsilon_n)} G^R_{RL}(\varepsilon, \Omega_N) \right) \right\}, \tag{3.2.14}$$

where

$$\mathcal{Z} = \sum_N Z_N(\varepsilon) \left(e^{-\beta E_N} + e^{-\beta(E_{N+1} \varepsilon_N)} \right). \tag{3.2.15}$$

We remind the reader about the relationship between the energy of N and $N+1$ electrons on the dot

$$E_{N+1} = E_N + \delta_\mu, \tag{3.2.16}$$

where we have defined $\delta_\mu = \Omega_N - \mu$. We tune the gate voltage near the degeneracy point which means that the energy is the same for having N or $N+1$ electrons on the dot. All other electron numbers of the dot are exponentially suppressed, so these terms are neglected. The Green functions have 4 terms in total since each electron number N or $N+1$ has two possibilities that energy level ε_n is empty or filled. The retarded Green function of the dot can now be written

$$\begin{aligned} G^R(\varepsilon) &= \sum_n \frac{1}{\mathcal{Z}} \left\{ e^{-\beta\varepsilon_n} G^R_{RL}(\varepsilon, \Omega_{N-1}) \right. & (3.2.17) \\ &+ \left. (1 + e^{-\beta(\varepsilon_n + \delta_\mu)}) G^R_{RL}(\varepsilon, \Omega_N) + e^{-\beta\delta_\mu} G^R_{RL}(\varepsilon, \Omega_{N+1}) \right\}, & (3.2.18) \end{aligned}$$

where

$$\mathcal{Z} = \left(1 + e^{-\beta\varepsilon_n}\right)\left(1 + e^{-\beta\delta_\mu}\right). \tag{3.2.19}$$

The standard resonant level retarded Green functions is given by [38, 34]

$$G^R_{RL}(\varepsilon, \Omega_N) = \frac{1}{\varepsilon - \varepsilon_n - \Omega_N + i\Gamma_n} \tag{3.2.20}$$

where Γ_n is the the tunneling rate for the resonant level. Inserting this into eq. (3.2.18) we obtain the final expression for the retarded Green function

$$\begin{aligned} G^R(\varepsilon) &= \sum_n \frac{1}{Z} \left\{ \frac{e^{-\beta\varepsilon_n}}{\varepsilon - \varepsilon_n - \Omega_{N-1} + i\Gamma_n} \right. \\ &+ \left. \frac{1 + e^{-\beta(\varepsilon + \delta_\mu}}{\varepsilon - \varepsilon_n - \Omega_N + i\Gamma_n} + \frac{e^{-\beta\delta_\mu}}{\varepsilon - \varepsilon_n - \Omega_{N+1} + i\Gamma_n} \right\}. \end{aligned} \tag{3.2.21}$$

Now we can do the same calculation for G^A. Inserting the result into the standard definition of TDoS, we obtain

$$\begin{aligned}\nu(\varepsilon) &= \frac{i}{2\pi}\left(G^R(\varepsilon) - G^A(\varepsilon)\right) \\ &= \frac{1}{\pi}\sum_n \frac{1}{Z}\left\{\frac{\Gamma_n e^{-\beta\varepsilon_n}}{(\varepsilon-\varepsilon_n-\Omega_{N-1})^2+\Gamma_n^2}\right. \\ &+ \left.\frac{\Gamma_n(1+e^{-\beta(\varepsilon_n+\delta_\mu)})}{(\varepsilon-\varepsilon_n-\Omega_N)^2+\Gamma_n^2} + \frac{\Gamma_n e^{-\beta\delta_\mu}}{(\varepsilon-\varepsilon_n-\Omega_{N+1})^2+\Gamma_n^2}\right\}.\end{aligned} \tag{3.2.22}$$

In the limit we are interested in, $\Gamma_n \ll \Delta \ll T \ll E_c$, we are allowed to approximate the Lorentzian with a delta function,

$$\frac{1}{\pi}\frac{\Gamma_n}{(\varepsilon-\varepsilon_n-\Omega_N)^2+\Gamma_n^2} = \sum_n \delta(\varepsilon-\varepsilon_n-\Omega_N). \tag{3.2.23}$$

Performing the sum over n and taking temperature smearing into account we can substitute the sum over the delta functions with the TDoS of non-interacting electrons, ν_0, which is valid when the mean level spacing, Δ, is much smaller then the temperature. Putting it all together we can now write the TDoS in the form

$$\frac{\nu(\varepsilon)}{\nu_0} = \frac{1}{1+e^{-\beta\delta_\mu}}\left\{\frac{e^{-\beta(\varepsilon-\Omega_{N-1})}}{1+e^{-\beta(\varepsilon-\Omega_{N-1})}} + \frac{1+e^{-\beta(\varepsilon-\Omega_N+\delta_\mu)}}{1+e^{-\beta(\varepsilon-\Omega_N)}} + \frac{e^{-\beta\delta_\mu}}{1+e^{-\beta(\varepsilon-\Omega_{N+1})}}\right\}. \tag{3.2.24}$$

Using the definition of the Fermi-function we can rewrite this in the more compact from

$$\frac{\nu(\varepsilon)}{\nu_0} = \frac{U(\varepsilon-\Omega_N) + e^{-\beta\delta_\mu}U(\varepsilon-\Omega_{N+1})}{1+e^{-\beta\delta_\mu}}, \tag{3.2.25}$$

where

$$U(\varepsilon-\Omega_N) = f(\varepsilon-\Omega_{N-1}) + 1 - f(\varepsilon-\Omega_N). \tag{3.2.26}$$

Comparing the result to the one obtained by Sedlmayr *et al.* [1], eq. (3.1.35), we see that we have successfully reproduced the correct result of the TDoS in the vicinity of the peak of conductance. We will now use this technique to calculate the noise in a Coulomb blockaded quantum dot in the next chapter.

3.3 Conclusion

In this chapter we have developed a new method to calculate the TDoS. We have compared to and found a complete agreement with the result rigorously derived by Sedlmayr *et al.* [1]. We have now built the foundation necessary to tackle the much harder question about noise in a Coulomb blockaded quantum dot.

Chapter 4

NOISE IN A COULOMB BLOCKADED QUANTUM DOT

We have now reached the core chapter of this thesis where we will calculate the noise power spectrum in a Coulomb blockaded quantum dot. As we discussed in the introduction and in the previous chapter there are a number of different parameters that describe the quantum dot. The key parameters that are of interest in this chapter are the charging energy E_c, temperature T, applied bias $eV = \mu_1 - \mu_2$, level spacing Δ and level width $\Gamma = \Gamma_1 + \Gamma_2$ of the quantum dot and the tunneling rates $\Gamma_\alpha = \pi\nu_0|t_\alpha|^2$. All these parameters give a large number of different regimes and some of them have already been studied. We will start by discussing the current state of the literature on this topic.

In the zero bias regime the noise follows the fluctuation dissipation theorem, $S(0) = 4kTG$, where G is the linear conductance. In the coming examples we set $T = 0$ and will focus on the dimensionless Fano factor in the zero frequency limit, $f = S(0)/2eI$. Without interactions the noise follows Poisson statistics and the Fano factor is 1. In the large bias limit, $E_c \ll eV$, the quantum dot can be viewed as two tunneling junctions in series. In this limit the Fano factor is given by [41, 42, 43]

$$f = \frac{G_1^2 + G_2^2}{G^2}, \tag{4.0.1}$$

where $G = G_1 + G_2$ with $G_\alpha = \pi e^2 \nu_0 \nu_d |t_\alpha|^2$, with ν_0 the bare density of states of the leads and ν_D the density of the dot states. This implies that the noise is suppressed below the Poisson statistics value and we have a Fano factor in the interval, $1/2 \leq f < 1$, with the minimum for a symmetric coupling to the leads, $G_1 = G_2$. The low bias regime $\Delta \ll eV \ll E_c$ is normally treated using the so called single particle "orthodox" theory. It is a method based on a classical master equation approach. The assumption necessary for this approach is that $\Delta \ll eV$. The Fano factor in this limit at the peak of conductance is given by [42, 43, 44, 45, 46]

$$f = \frac{\gamma_1^2 + \gamma_2^2}{\gamma^2}, \tag{4.0.2}$$

where $\gamma = \gamma_1 + \gamma_2$ and

$$\gamma_1 = \frac{G_1}{e^2} |\Delta_+(1, N)|, \tag{4.0.3}$$

$$\gamma_2 = \frac{G_2}{e^2} |\Delta_-(2, N+1)|, \tag{4.0.4}$$

with

$$\Delta_\pm(\alpha, N) = E(N \pm 1) - E(N) \mp \mu_\alpha. \tag{4.0.5}$$

So in our model, when we apply the bias voltage symmetrically around the dot, the Fano factors in eqs. (4.0.1) and (4.0.2) are identical. In the limit of very low bias $\Gamma \ll eV \ll \Delta$ the spectrum of the dot is now discrete. In this limit the single-particle picture still applies and a master equation approach is used in [47] to calculate the Fano factor

$$f = \frac{\Gamma_1^2 + \Gamma_2^2}{\Gamma^2}. \tag{4.0.6}$$

In this chapter we will calculate the noise in the Coulomb blockaded regime using a full quantum treatment of the problem in the Keldysh technique. The motivation for this is that around the peak of conductance the charging levels are degenerate and interacting. It is therefore not clear that the classical master equation approach will work for this situation. This work will give an answer whether or not the master equation approach gives a reasonable result and we will also be able to study the intermediate region in between the low bias regime and shot noise regime.

As a starting point we begin by calculating the current and the conductance. This will be useful for the checking the noise in some standard limits. Next we show why a naive extension of the method developed by Sedlmayr et al. [1] doesn't work for the noise. We then go on to calculate the noise using the new approach developed in section 3.2.

4.1 Current and conductance

We start from the universal Hamiltonian [14]

$$\mathcal{H} = \mathcal{H}_0 + \frac{E_c}{2}\left(\hat{N} - N_g\right)^2. \tag{4.1.1}$$

Here $\mathcal{H}_0$ is the non-interacting Hamiltonian of the electrons confined to the dot in a random potential. E_c is the charging energy of the dot, $\hat{N}$ is the number of electrons on the the dot and N_g is proportional to the gate voltage that is used to control the number of electrons of the dot. To be able to drive the system out of equilibrium we attach two non-interacting leads to the quantum dot and connect them through tunneling contacts

$$\mathcal{H}_t = \sum_{\alpha,k,n}\left(t_{\alpha n} c^\dagger_{\alpha,k} d_n + h.c.\right). \tag{4.1.2}$$

Here $c^{\dagger}_{\alpha,k}/c_{\alpha,k}$ creates/annihilates one electron with momentum k in lead $\alpha = 1, 2$ and $d^{\dagger}_{\alpha,k}/d_{\alpha,k}$ creates/anhiliates a electron on level n on the dot. The parameter $t_{\alpha n}$ is the tunneling rate and we assume that it is independent of the momentum of the electron, k and what energy level, n, involved in the tunneling.

In this chapter we are interested in the strong Coulomb blockaded regime. This means that the charging energy dominates over all other energy scales and we also require that the temperature is greater than the mean level spacing, Δ, which in turn is greater than the tunneling rate, Γ_α. To summarise we are in the regime $\Gamma_\alpha \ll \Delta \ll T \ll E_c$. When we start driving a current through the system we apply a bias voltage eV. In this chapter we will not put any restrictions on the bias voltage but we will mainly be interested in the shot noise regime, $T \ll eV$, which is the regime where thermal fluctuations are small and quantum fluctuations the most important.

The definition of the current is

$$I_\alpha = i \sum_{n,k} \left(t_{\alpha n} c^{\dagger}_{\alpha,k} d_n + h.c. \right). \tag{4.1.3}$$

In appendix A we derive the following standard expression for the current in the Keldysh technique from [74, 75],

$$I_\alpha = e\Gamma_\alpha \int_{-\infty}^{\infty} \frac{d\varepsilon}{4\pi} \left(Tr \left\{ G^K(\varepsilon) - (1 - 2f_\alpha(\varepsilon)) \left(G^R(\varepsilon) - G^A(\varepsilon) \right) \right\} \right), \tag{4.1.4}$$

where $\Gamma_\alpha = 2\pi\nu_\alpha |t_{\alpha n}|^2$ and ν_α is the bare density of states for the non-interacting lead electrons. We can now use the fact that we are in the strong Coulomb blockade regime where the dot is not allowed to accumulate charge. Since we only have two leads we have current

conservation, $I_1 + I_2 = 0$, which we use to remove the Keldysh component in eq. (4.1.4),

$$\begin{aligned} I &= e\frac{\Gamma_1\Gamma_2}{\Gamma}\int_{-\infty}^{\infty}\frac{d\varepsilon}{2\pi i}\left(f_2(\varepsilon)-f_1(\varepsilon)\right)\left(G^R(\varepsilon)-G^A(\varepsilon)\right) \\ &= \frac{e\nu_0\Gamma_1\Gamma_2}{\Gamma}\int_{-\infty}^{\infty}d\varepsilon\left(f_1(\varepsilon)-f_2(\varepsilon)\right)\frac{\nu(\varepsilon)}{\nu_0}, \end{aligned} \tag{4.1.5}$$

where we have introduced $\Gamma = \Gamma_1 + \Gamma_2$. In the linear response regime we can also calculate the differential conductance

$$G = \frac{dI}{dV} = e^2\nu_0\frac{\Gamma_1\Gamma_2}{\Gamma}\int_{-\infty}^{\infty}d\varepsilon\left(-\frac{\partial f(\varepsilon)}{\partial\varepsilon}\right)\frac{\nu(\varepsilon)}{\nu_0}. \tag{4.1.6}$$

At this stage in the calculation it is clear that we so far have not needed the new approach derived in the previous chapter since we can insert the result from either of the two eqs. (3.1.35) or (3.2.25) above and we obtain the result without problems. So it is time to take a closer look at where the old approach developed by Sedlmayr et al. [1] goes wrong when we calculate the noise.

4.2 Why the Sedlmayr-Yurkevich-Lerner approach to TDoS does not work for noise

In section 3.1 we calculated the isolated Green functions for a Coulomb blockaded quantum dot. To make the action quadratic in the fermionic fields we performed a Hubbard-Stratonovich transformation and made the fermionic fields quadratic at the expense of introducing a bosonic field. In the single particle Green function case this doesn't cause any problems. But in the case of the noise we also have two particle Green functions and in this case the bosonic field causes problems and the method becomes more or less impossible to use. The reason for this comes when we have introduced the bosonic field and we now have a quadratic action in the fermionic field. We can pull the integral over the bosonic fields out

front to use Wick's theorem and split the two particle Green function into two single particle Green functions. The problem that now occurs is that we don't have two independent single particle Green functions since the bosonic field couples the Green function together. More problematic is that this also cause all the 4 times to be connected and when we have to time order the 4 times over the Keldysh contour, the number of terms explodes. We also have the problems that the standard single particle Green function relations are not valid anymore. We now focus on using the new method we developed in chapter 3 to calculate the noise instead. In this approach we don't have to introduce the bosonic field to make the action quadratic and we can therefore circumvent all the problems caused by it.

4.3 The Model

We simplify the universal Hamiltonian of a quantum dot in eq. (4.1.1) by connecting the tunneling to only one level of the dot. This level is then removed from the universal Hamiltonian and treated separately. Putting this together lead to the following form of the Hamiltonian:

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}(E_{N'}) + \mathcal{H}_n + \mathcal{H}_0^{leads}, \tag{4.3.1}$$

where

$$\mathcal{H}_0 = \sum_{m \neq n} \xi d_m^\dagger(t) d_m(t), \tag{4.3.2}$$

is the kinetic part of the dot with level n removed. The charging term

$$\mathcal{H}(E_{N'}) = \frac{E_c}{2} \left(N' - N_g\right)^2, \tag{4.3.3}$$

only cares about the number of electrons on the dot and not their configuration. Finally the part of the Hamiltonian depending on n is given by

$$\mathcal{H}_n = \xi_n d_n^\dagger d_n(t) + \Omega(N') d_n^\dagger(t) d_n(t) + \sum_{\alpha,k} t_{\alpha,n} d_n^\dagger(t) c_{\alpha k}(t) + h.c. \tag{4.3.4}$$

4.4 Noise

The first step in the calculation of the noise is the same in the interacting case as for the resonant level in section 2.5. The resonant level result will also be part of the interacting system calculation since we will succeed in writing part of the noise as a resonant level model. We once again start from the current operator:

$$I_L = ie \sum_k \left[t_k c_k^\dagger d_n - t_k^* d_n^\dagger c_k \right]. \tag{4.4.1}$$

we suppose the index $\alpha = L$ in the following and suppress it. In terms of the current operator I_L the noise is now given by

$$\begin{aligned} S(t,t') &= \langle \{ \delta I_L(t), \delta I_L(t') \} \rangle \\ &= \langle \{ I_L(t), I_L(t') \} \rangle - 2 \langle I_L \rangle^2 \\ &= (ie)^2 \sum_{k,k'} \Big[t_k t_k' \langle c_k^\dagger(t) d_n(t) c_{k'}^\dagger(t') d_n(t') \rangle - t_k t_{k'}^* \langle c_k^\dagger d_n(t) d_n^\dagger(t') c_{k'}(t') \rangle \\ &- t_k^* t_{k'} \langle d_n^\dagger c_k(t) c_{k'}^\dagger(t) d_n(t') \rangle + t_k^* t_{k'}^* \langle d_n^\dagger(t) c_k(t) d_n^\dagger(t') c_{k'}(t') \rangle \Big] - 2 \langle I_L \rangle^2, \end{aligned} \tag{4.4.2}$$

where

$$\delta I = I - \langle I \rangle. \tag{4.4.3}$$

The next step is to define the two particle Green functions:

$$G_{1,nm}(t,t') = -\langle T c_k^\dagger(t) d_n(t) c_{k'}^\dagger(t') d_m(t')\rangle, \tag{4.4.4}$$

$$G_{2,nm}(t,t') = -\langle T c_k^\dagger(t) d_n(t) d_m(t')^\dagger c_{k'}(t')\rangle, \tag{4.4.5}$$

$$G_{3,nm}(t,t') = -\langle T d_n(t)^\dagger c_k(t) c_{k'}^\dagger(t') d_m(t')\rangle, \tag{4.4.6}$$

$$G_{4,nm}(t,t') = -\langle T d_n(t)^\dagger c_k(t) d_m(t')^\dagger c_{k'}(t')\rangle. \tag{4.4.7}$$

To be able to solve the Greens functions we need to make the approximation

$$\sum_{n,m} G_{nm}(t,t') = \sum_n G_{nn}(t,t'). \tag{4.4.8}$$

As we argued in section 3.2, this assumption is reasonable for a non-interacting system. We can now write the two particle Green functions in the same form as the single-particle Green function in section 3.2,

$$G_{nn}(t,t') = \frac{\left(Tr e^{-\beta H} c_k^\dagger(t) d_n(t) c_{k'}(t') d_n(t')\right)}{Tr\left(e^{-\beta H}\right)}. \tag{4.4.9}$$

We see that E_N once again provides weights to the different charging states of the quantum dot. Using the same method developed in the previous chapter we can now write the two particle Green's function in one resonant level part and one interacting part

$$G_{nn}(\varepsilon) = \frac{1}{\mathcal{Z}} \sum_N e^{-\beta E_N} Z_N(\varepsilon_n) G_{RL}(\varepsilon, \Omega_N) Z_{RL}, \tag{4.4.10}$$

where

$$\mathcal{Z} = \sum_N e^{-\beta E_N} Z_N Z_{RL}. \tag{4.4.11}$$

Using the same technique for all the Green functions, eqs. (4.4.4)-(4.4.7) we can rewrite the noise in the form:

$$S(0) = \frac{1}{\sum_N e^{-\beta E_N} Z_{RL}(\varepsilon_n, \Omega_N)} \sum_N e^{-\beta E_N} Z_{RL}(\varepsilon_n) S^{RL}(\varepsilon_n, \Omega_N). \tag{4.4.12}$$

This makes things much simpler since we can now use the results from section 2.5 about the resonant level noise with small modifications. Assuming that we are in the vicinity of the peak we can truncate the sum over N to the two terms closest to $N_g + 1/2$, and all other terms will be exponentially suppressed. After this truncation we can write the noise in the form

$$\begin{aligned} S(0) &= \sum_n \frac{1}{Z} \left\{ e^{-\beta\varepsilon_n} S^{RL}(\varepsilon_n, \Omega_{N-1}) + \left(1 + e^{-\beta(\varepsilon_n + \delta_\mu)}\right) S^{RL}(\varepsilon, \Omega_N) \right. \\ &+ \left. e^{-\beta\delta_\mu} S^{RL}(\varepsilon, \Omega_{N+1}) \right\}, \end{aligned} \tag{4.4.13}$$

where the partition function is now given by

$$Z = \left(1 + e^{-\beta\varepsilon_n}\right)\left(1 + e^{-\beta\Omega_N}\right). \tag{4.4.14}$$

Using the definition of the Fermi functions we can now write the noise in the much nicer form

$$\begin{aligned} S(0) &= \sum_n \frac{1}{1 + e^{-\beta\Omega_N}} \left\{ f(\varepsilon_n) S_{nRL}(0, \Omega_{N-1}) \right. \\ &+ \left. (1 - f(\varepsilon_n) + e^{-\beta\Omega_N} f(\varepsilon_n)) S_{nRL}(0, \Omega_N) + e^{-\beta\Omega_N}(1 - f(\varepsilon_n)) S_{nRL}(0, \Omega_{N+1}) \right\}. \end{aligned} \tag{4.4.15}$$

Here S_{nRL} is the standard result for the noise in a resonant level that was derived in section 2.5,

$$S_{nRL}(0, \Omega_N) = 2e^2 \int \frac{d\varepsilon}{2\pi} \left\{ [f_L(\varepsilon)(1 - f_L(\varepsilon)) + f_R(\varepsilon)(1 - f_R(\varepsilon))] \, T_n(\varepsilon, \Omega_N) \right.$$

$$+ \quad [f_L(\varepsilon) - f_R(\varepsilon)]^2 T_n(\varepsilon, \Omega_N)(1 - T_n(\varepsilon, \Omega_N))\Big\}. \tag{4.4.16}$$

In eq. (4.4.15) the strength of the method becomes clear since we can easily see the origin of each terms in the noise. In the first term we have the configuration of the dot that level n is occupied and we have $N-1$ electrons on the remaining levels of the dot. In the second term we have N electrons on the dot if we don't count level n, we therefore have two situations for level n, it can be either empty or filled. If it is filled we get a N+1 electrons on the dot and therefore this terms get the weight, $e^{-\beta\Omega_N}$. The empty n level get the weight 1. Finally we have an empty level n and $N+1$ electrons on the remaining of the levels. Now that we have a full expression for the noise we need to take a closer look at the tunneling rates to see if we can simplify the noise expression. We start with the linear term

$$\sum_n T_n(\varepsilon, \Omega_N) = \Gamma_1\Gamma_2 \sum_n G^R_{nRL}(\varepsilon, \Omega_N) G^A_{nRL}(\varepsilon, \Omega_N) \tag{4.4.17}$$

$$= \frac{\Gamma_1\Gamma_2}{\Gamma} 2\pi \sum_n \nu_{nRL}(\varepsilon, \Omega_N), \tag{4.4.18}$$

where

$$\sum_n \nu_{nRL}(\varepsilon, \Omega_N) = \frac{1}{\pi}\sum_n \frac{\Gamma/2}{(\varepsilon - \varepsilon_n - \Omega_N)^2 + (\Gamma/2)^2} = \sum_n \delta(\varepsilon - \varepsilon_n - \Omega_N) = \nu_0. \tag{4.4.19}$$

In this approximation we write the Lorentzian as a delta function which is valid, if Γ is small, $\Gamma \ll \Delta \ll T \ll E_c$. Substituting eq. (4.4.19) into eq. (4.4.18) we now find a simple form for the tunneling rate

$$\sum_n T_n(\varepsilon, \Omega_N) = 2\pi\nu_0 \frac{\Gamma_1\Gamma_2}{\Gamma}. \tag{4.4.20}$$

We now evaluate the quadratic term in a similar fashion

$$\sum_n (T_n(\varepsilon, \Omega_N))^2 = \left(\frac{\Gamma_1\Gamma_2}{\Gamma}\right)^2 4\pi^2 \sum_n \left(\nu_{nRL}(\varepsilon.\Omega_N)\right)^2, \tag{4.4.21}$$

Here the square density of states can be written in the form

$$\begin{aligned}\sum_n \left(\nu_{nRL}(\varepsilon.\Omega_N)\right)^2 &= \frac{1}{\pi^2}\sum_n \frac{(\Gamma/2)^2}{\left((\varepsilon-\varepsilon_n-\Omega_N)+(\Gamma/2)^2\right)^2} \\ &= -\left(\frac{\Gamma}{2}\right)^2 \frac{1}{\pi^2}\frac{d}{d(\Gamma/2)^2}\sum_n \frac{1}{(\varepsilon-\varepsilon_n-\Omega_N)^2+(\Gamma/2)^2} \\ &= -\left(\frac{\Gamma}{2}\right)^2 \frac{1}{\pi^2}\frac{d}{d(\Gamma/2)^2}\frac{2}{\Gamma}\sum_n \delta(\varepsilon-\varepsilon_n-\Omega_N) \\ &= \nu_0\frac{1}{\pi\Gamma}. \end{aligned} \tag{4.4.22}$$

Comparing this result to eq. (4.4.21) we otain the relation

$$\sum_n (T_n(\varepsilon,\Omega_N))^2 = 4\pi\nu_0\frac{(\Gamma_1\Gamma_2)^2}{\Gamma^3} = 2\frac{(\Gamma_1\Gamma_2)}{\Gamma^2}\sum_n T_n(\varepsilon,\Omega_N). \tag{4.4.23}$$

From this relation it is also clear why there are potential dangers in just expanding to t^2 order and throwing away all other terms. Since terms that appear to be of t^4 can in fact be of t^2 due to eq. (4.4.23). Inserting eqs. (4.4.16), (4.4.20) and (4.4.23) into, eq. (4.4.15), we obtain the final form of the noise we calculated in the previous chapter and where $U(\varepsilon-\Omega_N) = f(\varepsilon-\Omega_{N-1})+1-f(\varepsilon-\Omega_N)$.

$$\begin{aligned} S(0) &= 2e^2\nu_0\frac{\Gamma_1\Gamma_2}{\Gamma}\int d\varepsilon\frac{\nu(\varepsilon)}{\nu_0}\Big\{\left[f_L(\varepsilon)(1-f_L(\varepsilon))+f_R(\varepsilon)(1-f_R(\varepsilon))+(f_L(\varepsilon)-f_R(\varepsilon))^2\right] \\ &- 2\frac{\Gamma_1\Gamma_2}{\Gamma^2}\left[f_L(\varepsilon)-f_R(\varepsilon)\right]^2\Big\}. \end{aligned} \tag{4.4.24}$$

To obtain the tunneling density of states

$$\frac{\nu(\varepsilon)}{\nu_0} = \frac{U(\varepsilon-\Omega_N)+e^{-\beta(\Omega_N-\mu)}U(\varepsilon-\Omega_{N+1})}{1+e^{-\beta(\Omega_N-\mu)}}, \tag{4.4.25}$$

we use the delta functions $\sum_n \delta(\varepsilon-\varepsilon_n-\Omega_N)$ and combine them with the Fermi function depending on ε_n in eq. (4.4.15). The Fano factor, $f = S(0)/2eI$, can now be expressed in

the general case as

$$f = \frac{\int d\varepsilon \frac{\nu(\varepsilon)}{\nu_0} \left\{ f_L(\varepsilon)(1-f_L(\varepsilon)) + f_R(\varepsilon)(1-f_R(\varepsilon)) + (f_L(\varepsilon)-f_R(\varepsilon))^2 \left(\frac{\Gamma_1^2+\Gamma_2^2}{\Gamma^2} \right) \right\}}{\int d\varepsilon \frac{\nu(\varepsilon)}{\nu_0} (f_L(\varepsilon)-f_R(\varepsilon))}. \quad (4.4.26)$$

We will start by looking at the noise and the Fano factor in two simple limits $T = 0$ and $V = 0$. We start with $T = 0$ where the noise takes the simple form

$$S(0) = 2e^2\nu_0 \frac{\Gamma_1\Gamma_2}{\Gamma} \int d\varepsilon \frac{\nu(\varepsilon)}{\nu_0} \left\{ \left[(f_L(\varepsilon)-f_R(\varepsilon))^2 \left(\frac{\Gamma_1^2+\Gamma_2^2}{\Gamma^2} \right) \right] \right\}. \quad (4.4.27)$$

At zero temperature we can approximate the Fermi function with a unit step function and using that $(\theta(x)-\theta(y))^2 = \theta(x)-\theta(y)$. We can now write the Fano factor eq. (4.4.26) as:

$$f = \frac{\Gamma_1^2+\Gamma_2^2}{\Gamma^2}. \quad (4.4.28)$$

This is the standard result obtained in the shot noise limit at using the two state approximation in the orthodox theory [42, 43, 44, 45]. In this limit the noise is always suppressed below the Poisson value, $1/2 \leq f < 1$, with the maximal suppression taking place in the symmetric dot, i.e. $\Gamma_1 = \Gamma_2$. Next we look at the zero bias limit of the noise where we only have thermal noise. We start by writing the noise in this limit in the form

$$S(0) = 2e^2\nu_0 \frac{\Gamma_1\Gamma_2}{\Gamma} \int d\varepsilon \frac{\nu(\varepsilon)}{\nu_0} \left\{ [f_L(\varepsilon)(1-f_L(\varepsilon)) + f_R(\varepsilon)(1-f_R(\varepsilon))] \right\}. \quad (4.4.29)$$

When we applied the bias voltage we did it in the form $eV/2$ on the left and subtracted $eV/2$ on the right so that we can write the fermi functions as

$$f_L(\varepsilon) = \frac{1}{2} - \frac{1}{2}\tanh\left(\frac{\beta(\varepsilon-(\mu+eV/2))}{2} \right), \quad (4.4.30)$$

$$f_R(\varepsilon) = \frac{1}{2} - \frac{1}{2}\tanh\left(\frac{\beta(\varepsilon-(\mu-eV/2))}{2} \right). \quad (4.4.31)$$

The noise in equilibrium, i.e. when $eV = 0$, reduces to

$$\begin{aligned} S(0) &= e^2\nu_0\frac{\Gamma_1\Gamma_2}{\Gamma}\int d\varepsilon\frac{\nu(\varepsilon)}{\nu_0}\left(1-\tanh^2\left(\frac{\beta\varepsilon}{2}\right)\right) \\ &= \frac{4e^2}{\beta}\nu_0\frac{\Gamma_1\Gamma_2}{\Gamma}\int d\varepsilon\frac{\nu(\varepsilon)}{\nu_0}\left(-\frac{\partial f(\varepsilon)}{\partial\varepsilon}\right), \end{aligned} \tag{4.4.32}$$

which is exactly the fluctuation dissipation theorem, i.e. $S(0) = 4kTG$, where G is the conductance. We see that in these two simple limits the standard results are reproduced, we now take a look at the general case. Inserting eqs. (4.4.25), (4.4.30), and (4.4.31) into eq. (4.4.24) we obtain the full expression for the noise

$$\begin{aligned} S(0) &= 2e^2\nu_0\frac{\Gamma_1\Gamma_2}{\Gamma}\int d\varepsilon \\ &\times\left\{1+\frac{1}{2}\tanh\left(\frac{\beta\delta_\mu}{2}\right)\frac{\tanh\left(\frac{\beta\varepsilon}{2}\right)-\tanh\left(\frac{\beta\delta_\mu}{2}\right)}{1-\tanh\left(\frac{\beta\varepsilon}{2}\right)\tanh\left(\frac{\beta\delta_\mu}{2}\right)}\right. \\ &+\frac{1}{2(1+e^{-\beta\delta_\mu})}\left[\frac{\tanh\left(\frac{\beta\varepsilon}{2}\right)-\tanh\left(\frac{\beta(\delta_\mu+E_c)}{2}\right)}{1-\tanh\left(\frac{\beta\varepsilon}{2}\right)\tanh\left(\frac{\beta(\delta_\mu+E_c)}{2}\right)}\right. \\ &\left.\left.-e^{-\beta\delta_\mu}\frac{\tanh\left(\frac{\beta\varepsilon}{2}\right)-\tanh\left(\frac{\beta(\delta_\mu-E_c)}{2}\right)}{1-\tanh\left(\frac{\beta\varepsilon}{2}\right)\tanh\left(\frac{\beta(\delta_\mu-E_c)}{2}\right)}\right]\right\} \\ &\times\left\{\frac{1}{2}\left[1-\frac{\tanh^2\left(\frac{\beta\varepsilon}{2}\right)-\tanh^2\left(\frac{\beta eV}{4}\right)}{1-\tanh^2\left(\frac{\beta\varepsilon}{2}\right)\tanh^2\left(\frac{\beta eV}{4}\right)}\right]\right. \\ &\left.-\frac{\Gamma_1\Gamma_2}{\Gamma^2}\left[\tanh\left(\frac{\beta eV}{4}\right)\frac{1-\tanh^2\left(\frac{\beta\varepsilon}{2}\right)}{1-\tanh^2\left(\frac{\beta\varepsilon}{2}\right)\tanh^2\left(\frac{\beta eV}{4}\right)}\right]^2\right\}. \end{aligned} \tag{4.4.33}$$

This general expression is to complicated to simplify analytically, so we will investigate a couple of limits and then plot the Fano factor at the peak of conductance, $\delta_\mu = 0$. We start by writing the current by substituting $A = \tanh(\beta eV/4)$ and $x = \tanh(\beta\varepsilon/2)$. We can write the current in the form

$$I = e\frac{\Gamma_1\Gamma_2}{\Gamma}\int_{-1}^{1}dx\frac{\nu(x)}{\nu_0}\frac{A}{1-A^2x^2}, \tag{4.4.34}$$

where we have written the TDoS in the form

$$\begin{aligned}\frac{\nu(x)}{\nu_0} &= \left\{1+\frac{1}{2}\tanh\left(\frac{\beta\delta_\mu}{2}\right)\frac{x-\tanh\left(\frac{\beta\delta_\mu}{2}\right)}{1-x\tanh\left(\frac{\beta\delta_\mu}{2}\right)}\right. \\ &+ \left.\frac{1}{2(1+e^{-\beta\delta_\mu})}\left[\frac{x-\tanh\left(\frac{\beta(\delta_\mu+E_c)}{2}\right)}{1-x\tanh\left(\frac{\beta(\delta_\mu+E_c)}{2}\right)}-e^{-\beta\delta_\mu}\frac{x-\tanh\left(\frac{\beta(\delta_\mu-E_c)}{2}\right)}{1-x\tanh\left(\frac{\beta(\delta_\mu-E_c)}{2}\right)}\right]\right\}. \end{aligned} \tag{4.4.35}$$

We can now write the full expression for the noise in the form

$$\begin{aligned} S(0) &= 2\frac{e^2 2}{\beta}\nu_0\frac{\Gamma_1\Gamma_2}{\Gamma}\int_{-1}^{1}dx\frac{\nu(x)}{\nu_0} \\ &\times \left\{\frac{1}{2}\left[\frac{1+A^2}{1-x^2A^2}\right]-2\frac{\Gamma_1\Gamma_2}{\Gamma^2}\left[A^2\frac{1-x^2}{(1-x^2A^2)^2}\right]\right\}. \end{aligned} \tag{4.4.36}$$

4.4.1 At the peak of conductance

Our original motivation for using this full quantum treatment of the noise was to check if the master equation approach was this valid at degeneracy point where the charging levels are strongly correlated. We will in this section calculate the Fano factor at this point in the shot noise regime and compare this to the master equation result [47],

$$f=\frac{\Gamma_1^2+\Gamma_2^2}{\Gamma^2}. \tag{4.4.37}$$

At the degenarcy point, i.e. $\delta_\mu=0$, the TDoS take on the much simpler form

$$\frac{\nu(\varepsilon)}{\nu_0}=1+\frac{1}{2}\left[\tanh\left(\frac{\beta(\varepsilon-E_c)}{2}\right)-\tanh\left(\frac{\beta(\varepsilon+E_c)}{2}\right)\right]. \tag{4.4.38}$$

Inserting eq. (4.4.38) into eq. (4.4.36) we obtain the noise as

$$S(0) = 2\frac{e^2 2}{\beta}\nu_0\frac{\Gamma_1\Gamma_2}{\Gamma}\int_{-1}^{1}dx\left\{1+\frac{1}{2}\left(\frac{x-\tanh\left(\frac{\beta E_c}{2}\right)}{1-x\tanh\left(\frac{\beta E_c}{2}\right)}-\frac{x+\tanh\left(\frac{\beta E_c}{2}\right)}{1+x\tanh\left(\frac{\beta E_c}{2}\right)}\right)\right\}$$

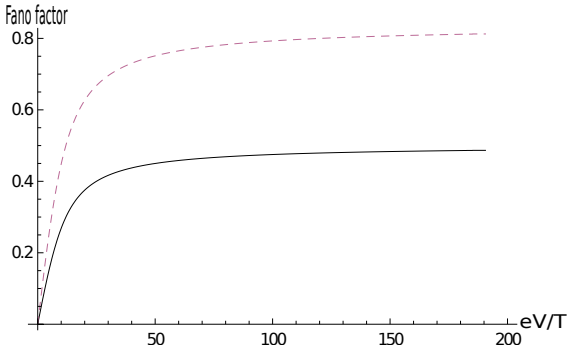


Figure 4.1: Fano factor at the peak of conductance with the thermal noise removed as a function of eV/T the lower curve is for $\Gamma_1 = \Gamma_2$ and the upper is $\Gamma_1 = 10\Gamma_2$. In both cases we have chosen the charging energy in units of temperature as, $E_c/T = 100$.

$$\times \quad \left\{ \frac{1}{2} \left[\frac{1+A^2}{1-x^2A^2} \right] - 2\frac{\Gamma_1\Gamma_2}{\Gamma^2} \left[A^2 \frac{1-x^2}{(1-x^2A^2)^2} \right] \right\}. \tag{4.4.39}$$

We start by plotting the Fano factor in Fig. 4.1 for different two different couplings to the leads. In the shot noise limit, $T \ll eV$, we obtain the expected $1/2$ for a symmetric coupling to the leads, $\Gamma_1 = \Gamma_2$. In the shot noise limit $A = \tanh(\beta eV/4) \to 1$ so we take the asymptotic limit and insert $A = 1$ in the expression for the noise and the current at the degeneracy point and we now see that the Fano factor in this point is

$$f = \frac{\Gamma_1^2 + \Gamma_2^2}{\Gamma^2}. \tag{4.4.40}$$

The master equation give the correct result also for these strongly correlated levels, so the master equation is valid both in the valley of conductance and at the peak of conductance in the shot noise regime. The strength of the full quantum treatment is that we can calculate the noise for any value of the bias voltage.

4.5 Conclusion

In this chapter we have used the method developed in chapter 3 to analytically calculate the noise in a Coulomb blockaded quantum dot. We obtain an exact solution valid in the

parameter range $\Delta \ll T \ll E_c$ as a function of distance to the peak and bias voltage. In equilibrium we recover the fluctuation dissipation theorem and at the degeneracy point the Fano factor is suppressed to 1/2 compared to the Poisson value (if the coupling to the leads are symmetric).

Chapter 5

BOSONISATION

We will in this chapter move to one-dimensional systems. One can easily realise without calculations that these are systems where interactions are important. The reason for this is simple, in one dimension as electrons cannot propagate through the system without pushing other electrons due to electron-electron interactions. Therefore no individual motion is possible and any individual excitation has to become a collective excitation. This is very different from higher dimensions where almost free quasi particle excitations are possible.

The early contributions to bosonisation were done by Tomonaga [48] and Luttinger [49] who introduced one of the first exactly solvable models in one dimension. It was later solved by Mattis and Lieb [50]. Other important contributions to the basic understanding of 1D systems was made by Dzyaloshinskii and Larkin [51], Efetov and Larkin [52] and Haldane [53]. The foundation of modern bosonisation, also known as the operator approach, started with the paper by Haldane in 1981 [53] where he proved rigourously how to construct Fermion creation-anihilation operators out of Bose ones. In this paper the concept of a Luttinger liquid was also coined and an interacting four-fermionic part was diagonalised in the bosonic description.

There are today several flavours of bosonisation and a number of good reviews [54, 55, 56, 57].

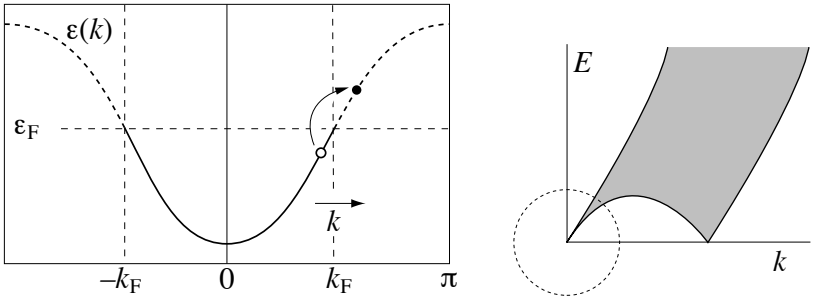


Figure 5.1: In the left figure we see an illustration of a simple particle hole excitation where we move one particle from below the Fermi energy to above. The excitation energy for this type of excitations is illustrated in the right hand figure and we see that for low energy and momentum the spectrum is linear. Taken from [55]

We will work with the version developed by von Delft and Scholler [54] also known as constructive bosonisation. In the remaining of this chapter we will start by diagonalising a generic interacting model of spinless fermions by rewriting them in terms of bosonic operators. This will provide a good introduction to why and how bosonisation work but to be able to calculate the correlation functions in the next chapter we also need to introduce and prove the identities for the fermionic operators described by bosonic operators.

The basic idea of bosonisation is that particle-hole excitations are bosonic in character and that most of the low lying excitations can be exhausted from these excitations. The reason for this is simple: If we take a look at Fig. 5.1 a particle-hole excitation is illustrated in the left hand figure and k is measure from the Fermi level, k_F. In the right of Fig. 5.1 wee see that this excitation has a linear spectrum. Because of this linear one-particle dispersion near the Fermi-level, the pairs have a narrow quasi-particle like dispersion near zero momentum, they can propagate coherently. This means that the particle and the hole have nearly the same group velocity and can propagate together. Any weak particle-hole attraction is bound to have a dramatic effect, i.e. bind the particles together into a coherently propagating entity: a new particle that will behave as a boson.

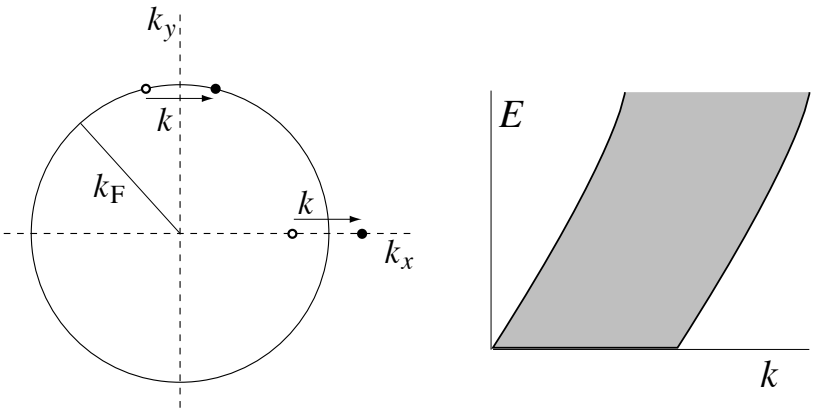


Figure 5.2: In two dimension the picture is very different and we see that it is possible create an excitation with momentum k that has a continuous spectrum of energies starting from zero. Taken from [55]

In higher dimension the situation is very different. For example in two dimensions we have a circular Fermi surface, illustrated in Fig. 5.2, which implies that a particle-hole pair with momentum k can have a continous spectrum of energies, starting from zero. Thus, the particle-hole spectrum is a continuum throughout and interactions have a harder time forming coherently propagating particle-hole pairs. All this implies that we can't construct a theory where we can rewrite fermionic fields as bosonic fields in 2D.

5.1 Some basic properties of fermions and bosons

Before we start introducing bosonisation we will start by introducing some basic properties of bosons and fermions that will be of importance later in this chapter.

The prerequisites that all the fermion creation and annihilation operators have to fulfil

to enable us to bosonise the theory, is the canonical anti-commutation relations

$$\{c_{k\eta}, c^{\dagger}_{k'\eta'}\} = \delta_{\eta\eta'}\delta_{kk'}, \tag{5.1.1}$$

where k is the unbounded momentum index, $k \in [\infty, \infty]$, and $\eta = 1, 2, \cdots M$ is the species index. The momentum index has the discrete form

$$k = \frac{2\pi}{L}\left(n_k - \frac{1}{2}\delta_b\right), \tag{5.1.2}$$

with $n_k \in \mathbb{Z}$, $\delta_b \in [0, 2)$ determines the boundary condition and L is the length of the system. Where the simplest cases are $\delta_b = 0, 1$ for periodic and anti-periodic boundary conditions. Starting from a given set of creation and annihilation operators that fulfils eqs. (5.1.1) and (5.1.2) we define the fermionic field operators as follows:

$$\psi_\eta(x) \equiv \left(\frac{2\pi}{L}\right)^{1/2} \sum_{k=-\infty}^{k=\infty} e^{-ikx} c_{k\eta}, \tag{5.1.3}$$

with the inverse

$$c_{k\eta} \equiv \frac{1}{(2\pi L)^{1/2}} \int_{-L/2}^{L/2} dx e^{ikx} \psi_\eta(x). \tag{5.1.4}$$

Given a set of given discrete k's, the field operators satisfy the boundary condition

$$\psi_\eta(x + L/2) = e^{i\delta_b}\psi(x - L/2), \tag{5.1.5}$$

The vacuum state $|0\rangle_0$ is defined as the state that fulfils the relation

$$\hat{c}_{k\eta}|\vec{0}\rangle_0 \equiv 0 \quad \text{for} \quad k > 0, \tag{5.1.6}$$

$$\hat{c}^{\dagger}_{k\eta}|\vec{0}\rangle_0 \equiv 0 \quad \text{for} \quad k < 0. \tag{5.1.7}$$

So all states up to the Fermi level are filled and all above are empty. A function of $c^\dagger_{k\eta}/c_{k\eta}$ operators is normal ordered, denoted : :, with respect to the vacuum state if we move all operators $c_{k\eta}$ with $k > 0$ and $c^\dagger_{k\eta}$ with $k < 0$ to the right of all other operators so that

$$: ABC\cdots :\equiv ABC\cdots - {}_0\langle\vec{0}|ABC\cdots|\vec{0}\rangle_0 \quad \text{for} \quad A, B, C \in \{c_{k\eta}; c^\dagger_{k\eta}\}. \tag{5.1.8}$$

We now define the number operator, N_η, that counts the number of electrons relative to the vacuum state

$$N_\eta \equiv \sum_{k=-\infty}^{\infty} : c^\dagger_{k\eta} c_{k\eta} := \sum_{k=-\infty}^{\infty} c^\dagger_{k\eta} c_{k\eta} - {}_0\langle\vec{0}|c^\dagger_{k\eta} c_{k\eta}|\vec{0}\rangle_0. \tag{5.1.9}$$

The set of states that have the same $\hat{N}_\eta$-eigenvalue, $\vec{N} = (N_1, N_2, \cdots, N_M) \in \mathbb{Z}$, is defined as the $\vec{N}$ particle Hilbert space, $\mathcal{H}_{\vec{N}}$. This space contains an infinite number of states that correspond to different configurations of particle-hole excitations. A general state in this space will be denoted $|\vec{N}\rangle$ while the ground state in this space is the state, $|\vec{N}\rangle_0$, that contains no particle-hole excitations and therefore is lowest energy state. This state is defined as follows:

$$|\vec{N}\rangle_0 = (C_1)^{N_1}(C_2)^{N_2}\cdots(C_M)^{N_M}|\vec{0}\rangle_0, \tag{5.1.10}$$

where

$$(C_N)^{N_\eta} = \begin{cases} c_{N_\eta,\eta} c_{N_\eta-1,\eta}\cdots c_{1,\eta} & \text{for} \quad N_\eta > 0 \\ 1 & \text{for} \quad N_\eta = 0 \\ c_{N_\eta,\eta} c_{N_\eta-1,\eta}\cdots c_{1,\eta} & \text{for} \quad N_\eta < 0 \end{cases}. \tag{5.1.11}$$

From this ground state we can create all other states through particle-hole excitations. For this purpose it is sufficient to consider the following bosonic creation and annihilation

operators.

$$\hat{b}^{\dagger}_{q\eta} \equiv \frac{i}{\sqrt{n_q}} \sum_k \hat{c}^{\dagger}_{k+q,\eta} \hat{c}_{k\eta}, \quad \hat{b}_{q\eta} \equiv \frac{-i}{\sqrt{n_q}} \sum_k \hat{c}^{\dagger}_{k-q,\eta} \hat{c}_{k\eta}, \quad q > 0. \tag{5.1.12}$$

We now have the necessary tools to prove some properties of the operators $b^{\dagger}_{\alpha q}/b_{\alpha q}$ and we start by the commutation relation for two operators on the same branch

$$[b_{q\eta}, b^{\dagger}_{q'\eta}] = \delta_{\eta\eta'} \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{n_q n_{q'}}} \left(c^{\dagger}_{k+q-q',\eta} c_{k\eta} - c^{\dagger}_{k+q,\eta} c_{k+q',\eta} \right). \tag{5.1.13}$$

We now have two cases $q = q'$ and $q \neq q'$. In the case when $q \neq q'$ both of the terms in eq. (5.1.13) are normal ordered and no subtleties can arise when we shift $k \to k - q'$ in the second term to cancel out the two terms. When q=q' we have to be more careful and first we have to normal order the two terms

$$\begin{aligned} [b_{q\eta}, b^{\dagger}_{q'\eta'}] &= \delta_{\eta\eta'}\delta_{qq'} \sum_{k=-\infty}^{\infty} \frac{1}{n_q} \left\{ \left(: c^{\dagger}_{k\eta} c_{k\eta} : - : c^{\dagger}_{k+q,\eta} c_{k+q,\eta} : \right) \right. \\ &\quad \left. + \left({}_0\langle\vec{0}| c^{\dagger}_{k\eta} c_{k\eta} |\vec{0}\rangle_0 - {}_0\langle\vec{0}| c^{\dagger}_{k+q,\eta} c_{k+q\eta} |\vec{0}\rangle_0 \right) \right\}. \end{aligned} \tag{5.1.14}$$

The first terms are now normal ordered so we can now cancel them out. From the definition of the vacuum in eqs. (5.1.6) and (5.1.7) we obtain

$$[b_{q\eta}, b^{\dagger}_{q',\eta'}] = \delta_{\eta\eta'}\delta_{qq'} \frac{1}{n_q} \left(\sum_{n_k=-\infty}^{0} - \sum_{n_k=-\infty}^{-n_q} \right) = \delta_{\eta\eta'}\delta_{qq'} \frac{1}{n_q} n_q = \delta_{\eta\eta'}\delta_{qq'}. \tag{5.1.15}$$

It is also straightforward to prove that the $\hat{b}_{q\eta}$ and $\hat{b}^{\dagger}_{q\eta}$ obey the remaining commutation relations

$$[\hat{b}_{q\eta}, \hat{b}_{q'\eta'}] = [\hat{b}^{\dagger}_{q\eta}, \hat{b}^{\dagger}_{q'\eta'}] = 0 \quad \text{for all} \quad q, q', \eta, \eta', \tag{5.1.16}$$

$$[\hat{\mathcal{N}}_{q\eta}, \hat{b}_{q'\eta'}] = [\hat{\mathcal{N}}_{q\eta}, \hat{b}^\dagger_{q'\eta'}] \quad \text{for all} \quad q, q', \eta, \eta', \tag{5.1.17}$$

where $\hat{\mathcal{N}}$ is the number operator.

From eq. (5.1.10) it is easy to verify that $|\vec{N}\rangle_0$ serves as a vacuum state for each $\vec{N}$ in $\vec{N}$-particle Hilbert space, $\mathcal{H}_{\vec{N}}$ for the bosonic excitations

$$b_{q\eta}|\vec{N}\rangle_0 = 0 \quad \text{for all} \quad q, \eta. \tag{5.1.18}$$

We can now define normal ordering also for bosonic operators

$$: ABC\cdots := ABC\cdots - {}_0\langle\vec{N}|ABC\cdots|\vec{N}\rangle_0 \quad \text{for} \quad A, B, C, \cdots \{b_{q\eta}, b^\dagger_{q\eta}\}. \tag{5.1.19}$$

It is obvious that every state in $\mathcal{H}_{\vec{N}}$ can be created by acting with a bilinear combination of fermion operator $|\vec{N}\rangle = f(c^\dagger_{k\eta}, c_{k\eta})|\vec{N}\rangle_0$. It is much les obvious that the same is true for bosonic operators but in fact there exist a function $f(b^\dagger)$ for every $|\vec{N}\rangle$ such that

$$|\vec{N}\rangle = f(b^\dagger)|\vec{N}\rangle_0. \tag{5.1.20}$$

We will not prove this highly non-trivial statement here but the interested reader can consult [54].

5.1.1 Bosonic fields

When we bosonise the fermionic operator $\psi_\eta(x)$ later in this chapter the bosonic field we will introduce in this chapter will be extremely useful. We start by defining the bosonic fields

$$\varphi_\eta(x) \equiv -\sum_{q>0} \frac{1}{\sqrt{n_q}} e^{-iqx} b_{q\eta} e^{-aq/2}, \tag{5.1.21}$$

$$\varphi_\eta^\dagger(x) \equiv -\sum_{q>0} \frac{1}{\sqrt{n_q}} e^{iqx} b_{q\eta}^\dagger e^{-aq/2}, \tag{5.1.22}$$

and their Hermitian combination

$$\phi_\eta \equiv \varphi_\eta(x) + \varphi_\eta^\dagger(x) = -\sum_{q>0} \frac{1}{\sqrt{n_q}} \left(e^{-iqx} b_{q\eta} + e^{iqx} b_{q\eta}^\dagger\right) e^{-aq/2}. \tag{5.1.23}$$

Here a is a mathematical regularisation parameter that is useful to prevent ultraviolet divergent momentum sums that occur in non-normal ordered expression and commutators. Using these definitions we can now write the normal-ordered electron densities using these bosonic operators

$$\rho_\eta(x) \equiv :\psi_\eta^\dagger(x)\psi_\eta(x): = \frac{2\pi}{L}\sum_q e^{-iqx} \sum_k :c_{k-q,\eta}^\dagger c_{k\eta}: \tag{5.1.24}$$

$$= \frac{2\pi}{L}\sum_{q>0} i\sqrt{n_q}\left(e^{-iqx} b_{q\eta} - e^{iqx} b_{q\eta}^\dagger\right) + \frac{2\pi}{L}\sum_k :c_{k\eta}^\dagger c_{k\eta}: \tag{5.1.25}$$

$$= \partial_x \phi_\eta(x) + \frac{2\pi}{L}\hat{N}_\eta, \tag{5.1.26}$$

where the last equality is valid if we insert the $a \to 0$ limit in eq. (5.1.23). The bosonic fields obey the commutation relations

$$[\varphi_\eta(x), \varphi_{\eta'}(x')] = [\varphi_\eta^\dagger(x), \varphi_{\eta'}^\dagger(x')] = 0, \tag{5.1.27}$$

and

$$[\varphi_\eta(x), \varphi_{\eta'}^\dagger(x')] = \delta_{\eta\eta'} \sum_{q>0} \frac{1}{n_q} e^{-q[i(x-x')+a]} \tag{5.1.28}$$

$$= -\delta_{\eta\eta'} \ln\left(1 - e^{-i\frac{2\pi}{L}[(x-x')-ia]}\right) \tag{5.1.29}$$

$$\to -\delta_{\eta\eta'} \ln\left(i\frac{2\pi}{L}[(x-x')-ia]\right), \tag{5.1.30}$$

where the last limit is valid when $L \to \infty$.

5.2 Luttinger model

It is now time to introduce the Luttinger model which is one of the basic and most important 1D models for interacting electrons

$$\mathcal{H} = \int dx \hat{\psi}^\dagger(x) \left(-\frac{1}{2m}\frac{\partial^2}{\partial x^2} - \mu \right) \psi(x) + \frac{1}{2}\int dx dx' V(x-x')\hat{\rho}(x)\hat{\rho}(x'). \tag{5.2.1}$$

Where $\hat{\psi}^\dagger/\hat{\psi}$ are the standard creation/annhiliation field operators, $\hat{\rho}(x) = \hat{\psi}^\dagger(x)\hat{\psi}(x)$ is the electron density operator and finally $V(x-x')$ describes the electron-electron interaction. In one dimension we have two Fermi points, $\pm k_F$, and near these two points the parabolic spectrum of our generic model can be described as linear

$$\varepsilon(k) = v_F(k - k_F), \qquad \text{around } k = k_F, \tag{5.2.2}$$

$$\varepsilon(k) = -v_F(k + k_F), \qquad \text{around } k = -k_F. \tag{5.2.3}$$

The first step towards the Luttinger model is to linearise the spectrum. This provides a good approximation around the Fermi points but we add an infinite number of energy states, see Fig. 5.3. Since the chemical potential is fixed this also means that we add an infinite number of particles to the model. However as long as the temperature is less then the Fermi energy the low lying excitations will be unaffected by states far away from the Fermi level. We will now introduce the index R and L to distinguish between electrons which live on the left or right moving branch. Right moving electrons count their momentum from the right Fermi point and left moving electrons count it from the left Fermi point

$$\hat{c}_{kR} = \hat{c}_{k_F+k}, \tag{5.2.4}$$

$$\hat{c}_{kL} = \hat{c}_{-k_F-k}. \tag{5.2.5}$$

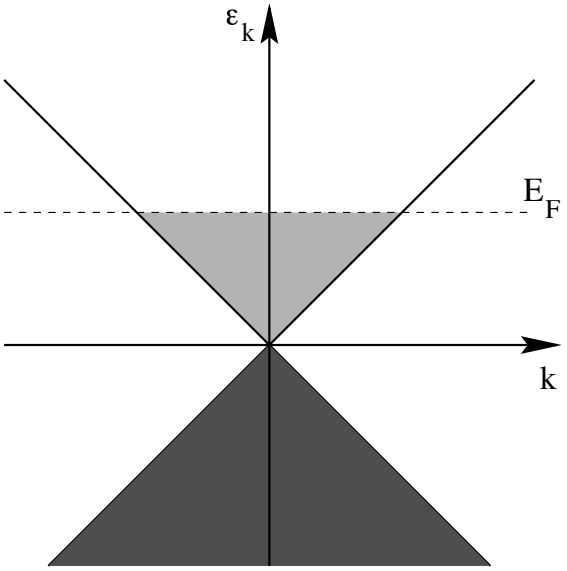


Figure 5.3: Illustration of the linearised spectrum of the Luttinger model. In the darkly shaded area in the negative energy sector we see the infinite number of negative states that we added when we linearised the spectrum.

The original field operators can now be written

$$\hat{\psi}(x) = \hat{\psi}_R(x)e^{ik_F x} + \hat{\psi}_L(x)e^{-ik_F x}, \tag{5.2.6}$$

where

$$\hat{\psi}_R(x) = \sum_k \hat{c}_{kR}\frac{e^{ikx}}{\sqrt{L}}, \tag{5.2.7}$$

$$\hat{\psi}_L(x) = \sum_k \hat{c}_{kL}\frac{e^{-ikx}}{\sqrt{L}}, \tag{5.2.8}$$

and L is the length of the system. For the low lying excitations in the vicinity of the Fermi points the fields $\psi_{R/L}$ varies slowly on the scale of $1/k_F$. We can therefore throw away all the rapidly oscillating terms of the type $e^{\pm 2ik_F x}$ since they will only provide small corrections

when they are integrated. With this assumption we arrive at the Luttinger model

$$\begin{aligned}\mathcal{H} &= iv_F \sum_{\eta=R/L=\pm} \int dx \hat{\psi}^\dagger_\eta(x) \eta \frac{\partial}{\partial x} \psi_\eta(x) \\ &+ \frac{1}{2} \sum_{\eta,\eta'=R/L=\pm} \int dx dx' V(x-x') \hat{\rho}_\eta(x) \hat{\rho}_{\eta'}(x'),\end{aligned} \tag{5.2.9}$$

where $\rho_\eta = \hat{\psi}^\dagger_\eta(x)\hat{\psi}_\eta(x)$, $\eta = R/L = \pm$.

5.3 Diagonalisation of the Luttinger model

We will now illustrate that we use bosonic operators to rewrite the interacting electron Hamiltonian as a quadratic bosonic Hamiltonian. We start by writing the Luttinger model in the form $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{int}$ where $\mathcal{H}_0$ is the Kinetic part

$$\mathcal{H}_0 = v_F \sum_{k,\eta} k \hat{c}^\dagger_{k\eta} \hat{c}_{k\eta}, \tag{5.3.1}$$

and $\mathcal{H}_{int}$ is the interacting part

$$\begin{aligned}\mathcal{H}_{int} &= \frac{\pi}{L} \sum_{k,k,q} V(q) \Big\{ \hat{c}^\dagger_{k+q,R} \hat{c}_{kR} \hat{c}^\dagger_{k'R} \hat{c}_{k'+q,R} + \hat{c}^\dagger_{k+q,R} \hat{c}_{kR} \hat{c}^\dagger_{k'L} \hat{c}_{k'-q,L} \\ &+ \hat{c}^\dagger_{k-qL} \hat{c}_{kL} \hat{c}^\dagger_{k'R} \hat{c}_{k'+q,R} + \hat{c}^\dagger_{k-q,L} \hat{c}_{k,L} \hat{c}^\dagger_{k',L} \hat{c}_{k'-q,L} \Big\},\end{aligned} \tag{5.3.2}$$

of the Hamiltonian and $V(q) = (1/2\pi) \int V(x) e^{-iqx}$. Next step is to introduce the operators

$$\hat{b}^\dagger_{q\eta} \equiv \frac{i}{\sqrt{n_q}} \sum_k \hat{c}^\dagger_{k+q,\eta} \hat{c}_{k,\eta}, \quad q > 0 \tag{5.3.3}$$

$$\hat{b}_{q,\eta} \equiv -\frac{i}{\sqrt{n_q}} \sum_k \hat{c}^\dagger_{k-q,\eta} \hat{c}_{k,\eta}, \quad q > 0, \tag{5.3.4}$$

where $q = 2\pi n/L$ and n_q is an integer. In these operators the interacting part of the Hamiltonian is quadratic

$$\mathcal{H}_{int} = \frac{2\pi}{L}\sum_{q>0} V(q) n_q \left(\hat{b}^\dagger_{qR}\hat{b}_{qR} + \hat{b}^\dagger_{Lq}\hat{b}_{qL} - \hat{b}^\dagger_{qR}\hat{b}^\dagger_{qL} - \hat{b}_{qL}\hat{b}_{qR}\right) + \frac{\pi V(q=0)}{L}\left(N_L + N_R\right)^2, \tag{5.3.5}$$

where N_L and N_R are the number of left and right moving electrons. We have also omitted the operator independent constant in the Hamiltonian. The fact that the Hamiltonian is quadratic in these new operators is encouraging.This shows that the interacting part of the Hamiltonian can be written in terms of bosonic operators that make the Hamiltonian quadratic. The remaining question is whether we can write the kinetic part of the Hamiltonian in these operators. The starting point is the Hamiltonian

$$\mathcal{H}_0 = \sum_\eta \mathcal{H}_{\eta 0}, \tag{5.3.6}$$

where the Hamiltonian on each branch is

$$\mathcal{H}_{\eta 0} = \sum_{k=-\infty}^{\infty} k c^\dagger_{k\eta} c_{k\eta}. \tag{5.3.7}$$

Since $[\mathcal{H}_{0\eta}, \hat{N}_{\eta'}] = 0$ for all η, η' every particle ground state is an eigenstate to $\mathcal{H}_{\eta 0}$, i.e. $\mathcal{H}_{\eta 0}|N\rangle_0 = E^N_{\eta 0}|N\rangle_0$. By simple inspection we see that the eigenvalue in the ground state is

$$\begin{aligned} E^N_{\alpha 0} = \langle N|\mathcal{H}_{\alpha 0}|N\rangle_0 &= \frac{2\pi}{L}\begin{cases} \sum_{n=1}^{N_\eta}(n - \delta_b/2) = \frac{1}{2}N_\eta^2 + \frac{1}{2}N_\eta(1-\delta_b) & \text{if} \quad N_\eta > 0 \\ \sum_{n=1}^{N_\eta} -(n-\delta_b/2) = \frac{1}{2}N_\eta^2 + \frac{1}{2}|N_\eta|(1-\delta_b) & \text{if} \quad N_\eta < 0 \end{cases} \\ &= \frac{\pi}{L}N_\eta(N_\eta + 1 - \delta_b). \end{aligned} \tag{5.3.8}$$

We now note that the operator $b^\dagger_{q\eta}$ increases the energy of any eigenstate $|E_\eta\rangle$ by q units. This yields that

$$[\mathcal{H}_{\eta 0}, b^\dagger_{q\eta'}] = q b^\dagger_{q\eta}\delta_{\eta,\eta'}, \tag{5.3.9}$$

which implies

$$\mathcal{H}_{\eta 0} b^\dagger_{q\eta}|E_\eta\rangle = (E_\eta + q)|E_\eta\rangle. \tag{5.3.10}$$

Since the Hilbert space H_N is completely spanned by operators $b^\dagger_{q\eta}$ acting on the ground state $|N\rangle_0$ it follows that $H_{\eta 0}$ must have a representation purely in bosonic operators. From eqs. (5.3.8) and (5.3.10) it is clear that the only representation that fulfills both these conditions is

$$\mathcal{H}_{\eta 0} = \sum_{q>0} q b^\dagger_{q\eta} b_{q\eta} + \frac{\pi}{L} N_\eta (N_\eta + 1 - \delta_b). \tag{5.3.11}$$

We now finally have a Hamiltonian that only depends on bosonic operators and before we find a Bogoljubov transformation to make it quadratic we neglect the constant term in eq. (5.3.11) and the $V(q=0)$ since these terms only contribute to processes that don't exist in the pure Luttinger liquid. It is backscattering processes turns left movers into right movers and vice versa and this requires impurity scattering which we will treat later in this thesis. So the final form of the Hamiltonian for the Luttinger model is

$$\mathcal{H} = \frac{2\pi v_F}{L} \sum_{q>0} n_q \left[b^\dagger_{qR} b_{qR} + b^\dagger_{qL} b_{qL} + \frac{V(q)}{v_F} \left(b^\dagger_{qR} b_{qR} - b^\dagger_{qL} b_{qR} - b^\dagger_{qR} b_{qL} + b^\dagger_{qL} b_{qL} \right) \right]. \tag{5.3.12}$$

This Hamiltonian is quadratic in the boson operators $b^\dagger_{q\eta}/b_{q\eta}$ and we can diagonalise it by the following Bogoljubov transformation

$$B_{q\pm} = \frac{1}{\sqrt{8}}\left[\left(\frac{1}{\sqrt{g}} + \sqrt{g}\right)(b_{qL} \mp b_{qR}) \pm \left(\frac{1}{\sqrt{g}} - \sqrt{g}\right)\left(b^\dagger_{qL} \mp b^\dagger_{qR}\right)\right], \tag{5.3.13}$$

where $g = v_f/v$ and $v = v_F\sqrt{1 + 2V(q=0)/v_F}$. We can now write the Hamiltonian in terms of these new operators in the diagonal form

$$\mathcal{H} = v\frac{2\pi}{L}\sum_{\nu=\pm}\sum_{q>0} n_q \hat{B}^\dagger_{q\nu}\hat{B}_{q\nu}. \tag{5.3.14}$$

To summarise we have showed that it is possible to take a strongly interacting electron model and rewrite it as a non-interacting bosonic theory.

5.4 Bosonisation identities

In the previous section we diagonalised the Tomonaga-Luttinger model, but to be able to calculate correlation functions we have to be able to express the Fermionic creation/anhiliation operators in terms of bosonic ones. From the definition of the operators

$$\psi_\eta(x) \equiv \frac{2\pi}{L}\sum_{k=-\infty}^{\infty} e^{-ikx} c_{k\eta}, \tag{5.4.1}$$

and

$$b_{q\eta} \equiv -\frac{i}{\sqrt{n_q}}\sum_{k=-\infty}^{\infty} c^\dagger_{k-q,\eta}c_{k\eta}, \tag{5.4.2}$$

it follows that

$$[b_{q\eta'}, \psi_\eta(x)] \;=\; \delta_{\eta\eta'}\alpha_\eta\psi_\eta(x), \tag{5.4.3}$$

$$\left[b^{\dagger}_{\eta' q}, \psi_{\eta}(x)\right] \;=\; \delta_{\eta\eta'}\alpha^{*}_{\eta}\psi_{\eta}(x), \tag{5.4.4}$$

where $\alpha_q = \frac{i}{\sqrt{n_q}} e^{iqx}$. From the definition of the N-particle ground state, $|N\rangle_0$, it follows that $b_{q\eta}|N\rangle_0 = 0$. Starting from the commutation relation eq. (5.4.3) we see that

$$\left[b_{q\eta'}, \psi_{\eta}(x)\right]|N\rangle_o = b_{q\eta'}\psi_{\eta}|N\rangle_0 = \delta_{\eta\eta'}\alpha_q\psi_{\eta}(x)|N\rangle_0, \tag{5.4.5}$$

from which it follows that $\psi_\eta(x)|N\rangle_0$ is an eigenstate of $b_{q\eta}$ with the eigenvalue α_q. This implies that $\psi_\eta(x)|N\rangle_0$ has an coherent state expansion in the form [58]

$$\psi_{\eta}(x)|N\rangle_0 = \exp\left(\sum_{q>0}\alpha_q(x)b^{\dagger}_{q\eta}\right)F_{\eta}\hat{\lambda}_{\eta}|N\rangle_0 = e^{-i\varphi^{\dagger}(x)}F_{\eta}\hat{\lambda}_{\eta}|N\rangle_0. \tag{5.4.6}$$

Where we have introduced Klein factors, F_η, that satisfy the commutation relations

$$[b_{q\eta}, F^{\dagger}_{\eta'}] = [b^{\dagger}_{q\eta}, F^{\dagger}_{\eta'}] = [b_{q\eta}, F_{\eta'}] = [b^{\dagger}_{q\eta}, F_{\eta'}] = 0 \quad \text{for all } \eta, \eta', q. \tag{5.4.7}$$

Taking a careful look at what this actually implies, one can start by writing $\psi_\eta(x)$ in the Fourier expansion form

$$\psi_{\eta}(x) = \left(\frac{2\pi}{L}\right)^{1/2}\sum_{k} e^{-ikx}c_{k\eta}. \tag{5.4.8}$$

Applying this to the state $|N\rangle_0$ we create a infinite linear combination of states $\psi_\eta(x) = \left(\frac{2\pi}{L}\right)^{1/2}\sum_k e^{-ikx}c_{k\eta}|N\rangle_0$. In the right hand side of eq. (5.4.6) F_η removes the top η-electron to form the new state $c_{N_\eta k}|N\rangle_0$ and we form an infinite linear combination of states by operating with $e^{-i\varphi^{\dagger}(x)}$. The statement that these set of states are equivalent is highly non-trivial since naively one would expect that the exponential $e^{-i\varphi^{\dagger}(x)}$ would create a much larger set of states. However exploiting the properties of coherent states eq. (5.4.6) guarantees that of all the combinations of particle-hole states combined in $e^{-i\varphi_n(x)}$ only those states contribute,

when acting on $c_{N_\eta\eta}|N\rangle_0$, that fills its empty N_η-level by moving to the later a single η-electron from a lover filled state. Remarkably all other combinations (that would leave a η-electron above N_η) cancel out to zero. To evaluate the operator $\hat{\lambda}_\eta$ we calculate the expectation value

$$ {}_0\langle N|F_\eta^\dagger\psi_\eta(x)|N\rangle_0 = \lambda_\eta(x), \tag{5.4.9}$$

where we have used eq. (5.4.6) for $\psi_\eta(x)$, commuted $e^{-i\varphi^\dagger(x)}$ past F_η and finally used ${}_0\langle N|e^{-i\varphi^\dagger(x)} = {}_0\langle N|$ by eq. (5.1.18). If we insted use the Fourier expansion of $\psi_\eta(x)$ and insert it into eq. (5.4.9) we note that neither $|N\rangle_0$ nor ${}_0\langle N|F_\eta^\dagger$ contain any particle-hole pairs, we realise that only terms in the sum with $n_k = N_k$ can contribute, i.e. $k = \frac{2\pi}{L}(N_\eta - \frac{1}{2}\delta_b)$,

$$\langle N|F_\eta^\dagger\psi_\eta(x)|N\rangle_0 = \left(\frac{2\pi}{L}\right)^{1/2} e^{-i\frac{2\pi}{L}(N_\eta-\frac{1}{2}\delta_b)x}. \tag{5.4.10}$$

So we can conclude

$$\lambda_\eta(x) = \left(\frac{2\pi}{L}\right)^{1/2} e^{-i\frac{2\pi}{L}(N_\eta-\frac{1}{2}\delta_b)x}. \tag{5.4.11}$$

The next step in the derivation of the Bosonisation indentities is to investigate the action of ψ_η on a arbritary state $|N\rangle$ in the Fock space which by eq. (5.1.20) we can write as $|N\rangle = f(\{b^\dagger_{q\eta'}\})|N\rangle$. Before we start we introduce two very useful idenities

$$\begin{aligned} \psi_\eta f(\{b^\dagger_{q\eta'}\}) &= f(\{b^\dagger_{q\eta'} - \delta_{\eta\eta'}\alpha^*_q(x)\})\psi_\eta(x), & (5.4.12)\\ f(\{b^\dagger_{q\eta'} - \delta_{\eta\eta'}\alpha^*_q(x)\}) &= e^{-i\varphi(x)} f(\{b^\dagger_{q\eta'}\})e^{i\varphi(x)}. & (5.4.13) \end{aligned}$$

We start by applying the fermionic field operator on a arbitrary state, $|N\rangle$, in Fock space and then we use eq. (5.1.20) to obtain:

$$\begin{aligned}
\psi_\eta(x)|N\rangle &= \psi_\eta(x) f(\{b^\dagger_{q\eta'}\})|N\rangle_0 && (5.4.14)\\
&= f(\{b^\dagger_{q\eta'} - \delta_{\eta\eta'}\alpha^*_q(x)\})\psi_\eta(x)|N\rangle_0 && (5.4.15)\\
&= f(\{b^\dagger_{q\eta'} - \delta_{\eta\eta'}\alpha^*_q(x)\})e^{-i\varphi^\dagger_\eta(x)}F_\eta\hat{\lambda}_\eta|N\rangle_0, && (5.4.16)
\end{aligned}$$

where eq. (5.4.15) follows by inserting eq. (5.4.12) into eq. (5.4.14) and then we obtain eq. (5.4.16) from eq. (5.4.6). We now use the commutation relations for the Klein factors eq. (5.4.7) to move the Klein factor up front:

$$\begin{aligned}
\psi_\eta(x)|N\rangle &= F_\eta\hat{\lambda}_\eta e^{-i\varphi^\dagger_\eta(x)} f(\{b^\dagger_{q\eta'} - \delta_{\eta\eta'}\alpha^*_q(x)\})|N\rangle_0 && (5.4.17)\\
&= F_\eta\hat{\lambda}_\eta e^{-i\varphi^\dagger_\eta(x)}e^{-i\varphi_\eta(x)} f(\{b^\dagger_{q\eta'}\})e^{i\varphi_\eta(x)}|N\rangle_0 && (5.4.18)\\
&= F_\eta\hat{\lambda}_\eta e^{-i\varphi^\dagger_\eta(x)}e^{-i\varphi_\eta(x)} f(\{b^\dagger_{q\eta'}\})|N\rangle_0 && (5.4.19)\\
&= F_\eta\hat{\lambda}_\eta e^{-i\varphi^\dagger_\eta(x)}e^{-i\varphi_\eta(x)}|N\rangle. && (5.4.20)
\end{aligned}$$

Eq. (5.4.18) follows from eq. (5.4.13) and next we use the definition of the vacuum state, eq. (5.1.18), to see that eq. (5.4.18) can be rewritten into eq. (5.4.19). Finally eq. (5.4.20) follows from eq. (5.1.20). Since $|N\rangle$ is an arbitrary state in Fock space these formulas, also known as bosonisation formulas for ψ_η, hold as operator identities in Fock space valid for all L.

$$\begin{aligned}
\psi_\eta(x) &= F_\eta\hat{\lambda}_\eta e^{-i\varphi^\dagger_\eta(x)}e^{-i\varphi_\eta(x)} && (5.4.21)\\
&= F_\eta\left(\frac{2\pi}{L}\right)^{1/2} e^{-i\frac{2\pi}{L}(\hat{N}_\eta - \frac{1}{2}\delta_b)x}e^{-i\varphi^\dagger_\eta(x)}e^{-i\varphi_\eta(x)} && (5.4.22)\\
&= F_\eta a^{-1/2}e^{-i\frac{2\pi}{L}(\hat{N}_\eta - \frac{1}{2}\delta_b)x}e^{-i\phi_\eta(x)} && (5.4.23)\\
&= F_\eta a^{-1/2}e^{-i\Phi_\eta(x)}. && (5.4.24)
\end{aligned}$$

Where eq. (5.4.22) follows from eq. (5.4.11). To show eq. (5.4.23) we need to use the Baker-Hausdorff formula from which it follows

$$e^{-i\varphi_\eta^\dagger(x)}e^{-i\varphi_\eta(x)} = e^{-i(\varphi_\eta^\dagger+\varphi_\eta)(x)}e^{-[i\varphi_\eta(x),i\varphi_\eta^\dagger(x)]} = \left(\frac{L}{2\pi a}\right)^2 e^{-i\phi(x)}. \tag{5.4.25}$$

using the commutations relation eq. (5.1.29). This concludes the derivation of the Bosonisation identities and they are all equivalent. The strength of the constructive bosonisation approach is that since we derived the formulas step by step from first principles there is no need to check their validity by calculating correlators ,$\langle\psi_\eta\psi_\eta^\dagger\rangle$ or anti-commutators, $\{\psi_\eta, \psi_\eta^\dagger\}$.

5.5 Summary

In this chapter we have studied the basic properties of electrons and bosons in one dimension. The first main result we derived was the introduction of the Luttinger model a strongly interacting electron model which can be rewritten using bosonic operators as a non-interacting bosonic theory. The main achievement was the derivation of the bosonic identities. This method of writing fermionic creation and annihilation operators in terms of bosonic ones will be used in the chapter to come.

Chapter 6

NOISE IN KONDO SYSTEMS

In this chapter we will discuss noise in Kondo systems. We start by familiarising ourselves with Kondo systems and how to deal with impurities in one dimension by studying the Schiller and Hershfield [59] solution of a non-equilibrium Kondo system. We start by mapping this system on to a solvable resonant level model. After we find a solvable model we discuss how the mapping of the original system affects the noise calculations. Since much of the principles involved in calculating the noise in this system is similar to the Coulomb dot we focus on discussing how to calculate the noise and the results rather then giving all the details. The next step is to discuss the results of Sela et al. [60] and the generalisation of this result by Fujii [61]. Finally we discuss a new system that would be interesting to generalise these results to.

6.1 The solvable model

We will in this section reproduce the derivation by Schiller and Hersfield [59] of a solvable non-equilibrium Kondo model in the Toulouse limit. The starting point is a spin 1/2 impurity that we place in between two one dimensional non-interacting leads that are attached via tunneling. The one dimensional fields, $\psi_{\alpha\sigma}$, interact with the impurity via the conduction-

electron spin densities

$$\vec{S}_{\alpha\beta} = \frac{1}{2}\sum_{\sigma\sigma'} \psi^{\dagger}_{\alpha\sigma}\vec{\sigma}_{\sigma\sigma'}\psi_{\beta\sigma'}. \tag{6.1.1}$$

The operators S_{LL} and S_{RR} are independent spin densities of the left and rights leads while S_{RL} and S_{LR} induce tunneling in between the leads. The full Hamiltonian of the system has the form:

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_B + \mathcal{Y}_0 + \mathcal{H}_K, \tag{6.1.2}$$

where

$$\begin{aligned}
\mathcal{H}_0 &= i\sum_{\alpha=R,L}\sum_{\sigma=\uparrow,\downarrow}\int dx \psi^{\dagger}_{\alpha\sigma}(x)\partial_x\psi_{\alpha\sigma}, && (6.1.3)\\
\mathcal{H}_B &= -h\tau^z, && (6.1.4)\\
\mathcal{Y}_0 &= \frac{V}{2}\sum_{\sigma=\uparrow,\downarrow}\int dx \left[\psi^{\dagger}_{L\sigma}(x)\psi_{L\sigma}(x) - \psi^{\dagger}_{R\sigma}(x)\psi_{R\sigma}(x)\right], && (6.1.5)\\
\mathcal{H}_K &= \sum_{\alpha\beta=L,R}\sum_{\nu=x,y,z} J_\nu S^{\nu}_{\alpha\beta}\tau^{\nu}, && (6.1.6)
\end{aligned}$$

where $\vec{\tau} = \vec{\sigma}/2$ and $\vec{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$ is the vector of Pauli matrices

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6.1.7}$$

We have also introduce a magnetic field, h, and an applied bias voltage, V, to drive the system out of euilibrium. Schiller and Hershfield [59] showed that this model is solvable also for non-equilibrium situations if we choose the following set of parameters:

$$J^{\alpha\beta}_x = J^{\alpha\beta}_y = J^{\alpha\beta}_\perp, \tag{6.1.8}$$

$$J^{LR}_\perp = J^{\alpha\beta}_\perp, \tag{6.1.9}$$

$$J_z^{LR} = J_z^{RL} = 0, \tag{6.1.10}$$

$$J_z^{LL} = J_z^{RR} = J_z = 2\pi. \tag{6.1.11}$$

Why this set of parameters, known as the Toulouse limit, is special will be clear as we continue diagonalising the Hamiltonian and this set of parameters will guarantee that the final Hamiltonian is quadratic. We can now rewrite the Kondo part of the Hamiltonian in the more convenient form

$$\mathcal{H}_K = \mathcal{H}_K^{||} + \mathcal{H}_K^{\perp}, \tag{6.1.12}$$

where

$$\begin{aligned} \mathcal{H}_K^{||} &= \frac{1}{2}\left[\Psi_\uparrow^\dagger(0)\hat{J}_z\Psi_\uparrow(0) - \Psi_\downarrow^\dagger(0)\hat{J}_z\Psi_\downarrow(0)\right]\tau^z, \\ \mathcal{H}_K^{\perp} &= \frac{1}{2}\Psi_\uparrow^\dagger(0)\hat{J}_\perp\Psi_\downarrow(0)\tau^- + \frac{1}{2}\Psi_\downarrow^\dagger(0)\hat{J}_\perp\Psi_\uparrow(0)\tau^+, \end{aligned} \tag{6.1.13}$$

and

$$\hat{J}_z = \begin{pmatrix} J_z^{RR} & J_z^{RL} \\ J_z^{LR} & J_z^{LL} \end{pmatrix}, \quad \hat{J}_\perp = \begin{pmatrix} J_\perp^{RR} & J_\perp^{RL} \\ J_\perp^{LR} & J_\perp^{LL} \end{pmatrix}. \tag{6.1.14}$$

We have also introduced the spinor notation

$$\Psi_\sigma(x) = \begin{pmatrix} \psi_{L\sigma}(x) \\ \psi_{R\sigma}(x) \end{pmatrix}, \tag{6.1.15}$$

and the standard raising and lowering operators for impurity spins, $\tau^\pm = \tau^x \pm i\tau^y$. The non-equilibrium term of the Hamiltonian, $\mathcal{Y}_0$, means that we have to be careful when we map this problem on to a non-interacting one. Normally this requires a canonical transformation that reduces the Hamiltonian to a quadratic form. In the non-equilibrium situation the transfor-

mation also has to preserve the quadratic form of $\mathcal{Y}_0$ otherwise the problem will remain a truly many-body problem. For our Hamiltonian we will use the bosonisation identities that we derived in the previous chapter. We will follow the tranformation introduced by Emery and Kivelson [62] and we start by introducing four bosonic fields $\Psi_{\alpha\sigma}$ where $\alpha = L, R$ and $\sigma = \uparrow, \downarrow$. Using eq. (5.4.24) we can now write the fermionic operators in the form

$$\psi_{\alpha\sigma} = \frac{e^{i\varphi_{\alpha\sigma}}}{2\pi a} e^{-i\Phi_{\alpha\sigma}(x)}, \tag{6.1.16}$$

where we have written the Klein factor in the form $F_{\alpha\sigma} = e^{i\varphi_{\alpha\sigma}}$ which is a standard notation in field theoretical bosonisation. Writing the Klein factor in this form can lead to some subtleties but we refer the reader to [54] for details. Driving the system out of equilibrium can also cause problems for the validity of the bosonisation identities since it can make the assumption that the spectrum is linear invalid. In our case this is not the case and we can safely use the bosonisation technique since we have the whole system at the same temperature. For an introduction to what happens if one starts to drive one dimensional systems out of the linear spectrum regime we recommend the work by Gutman et al. [63, 64, 65]. We choose the following phases of the Klein factors

$$\begin{aligned}
\varphi_{L\uparrow} &= \int_{-\infty}^{\infty} \left[\psi^\dagger_{L,\downarrow}\psi_{L\downarrow} + \psi^\dagger_{R,\uparrow}\psi_{R\uparrow} + \psi^\dagger_{L,\downarrow}\psi_{L\downarrow}\right] dx, && (6.1.17)\\
\varphi_{L\downarrow} &= \int_{-\infty}^{\infty} \left[\psi^\dagger_{R,\uparrow}\psi_{R\uparrow} + \psi^\dagger_{L,\downarrow}\psi_{L\downarrow}\right] dx, && (6.1.18)\\
\varphi_{R\uparrow} &= \int_{-\infty}^{\infty} \psi^\dagger_{L,\downarrow}\psi_{L\downarrow} dx, && (6.1.19)\\
\varphi_{R\downarrow} &= 0. && (6.1.20)
\end{aligned}$$

We also introduce new bosonic fields from the four bosonic field, $\Phi_{\alpha,\sigma}$,

$$\begin{aligned}
\Phi_c &= \frac{1}{2}\left(\Phi_{L\uparrow} + \Phi_{L\downarrow} + \Phi_{R\uparrow} + \Phi_{R\downarrow}\right), && (6.1.21)\\
\Phi_s &= \frac{1}{2}\left(\Phi_{L\uparrow} - \Phi_{L\downarrow} + \Phi_{R\uparrow} - \Phi_{R\downarrow}\right), && (6.1.22)
\end{aligned}$$

$$\Phi_f = \frac{1}{2}\left(\Phi_{L\uparrow} + \Phi_{L\downarrow} - \Phi_{R\uparrow} - \Phi_{R\downarrow}\right), \tag{6.1.23}$$

$$\Phi_{sf} = \frac{1}{2}\left(\Phi_{L\uparrow} - \Phi_{L\downarrow} - \Phi_{R\uparrow} + \Phi_{R\downarrow}\right), \tag{6.1.24}$$

where the new bosonic fields correspond to collective charge, spin, flavour and spin flavour respectively. Doing the rotation also in the Klein factor phase fields and writing them in terms of the bosonic fields [59] we obtain:

$$\varphi_c = \frac{1}{4}\int \left[3\nabla\Phi_c(x) - \nabla\Phi_s(x) - 2\nabla\Phi_f(x)\right] dx, \tag{6.1.25}$$

$$\varphi_s = \frac{1}{4}\int \left[\nabla\Phi_c(x) - \nabla\Phi_s(x)\right] dx, \tag{6.1.26}$$

$$\varphi_f = \frac{1}{4}\int \left[2\nabla\Phi_c(x) - \nabla\Phi_f(x) - \nabla\Phi_{sf}(x)\right] dx, \tag{6.1.27}$$

$$\varphi_{sf} = \frac{1}{4}\int \left[\nabla\Phi_f(x) - \nabla\Phi_{sf}(x)\right] dx. \tag{6.1.28}$$

It is now time to insert the bosonic identities 6.1.16 into the Hamiltonian and using the short hand notation $\chi_\nu = \Phi_\nu(0) - \varphi_\nu$ and writing the bosonic field in the charge, spin, flavour and spin flavour form we obtain the bosonic version of the Hamiltonian

$$\begin{aligned}
\mathcal{H} = & \frac{\hbar v_F}{4\pi}\sum_{\nu=c,s,f,fs}\int_{-\infty}^{\infty}\left(\nabla\Phi_\nu\right)^2 dx \\
& + \frac{J^+}{\pi a}\left[-\tau^x\sin(\chi_s) + \tau^y\cos(\chi_s)\right]\cos(\chi_{sf}) \\
& - \frac{J^-}{\pi a}\left[\tau^x\cos(\chi_s) + \tau^y\sin(\chi_s)\right]\sin(\chi_{sf}) \\
& - \frac{J_\perp^{LR}}{\pi a}\left[\tau^x\cos(\chi_s) + \tau^y\sin(\chi_s)\right]\sin(\chi_f) \\
& + \frac{J_z}{2\pi}\nabla\Phi_s(x)\tau^z - h\tau^z,
\end{aligned} \tag{6.1.29}$$

and

$$\mathcal{Y}_0 = \frac{eV}{2\pi}\int \nabla\Phi_f(x)dx, \tag{6.1.30}$$

where we have introduced

$$J^{\pm} = \frac{1}{2}\left(J_{\perp}^{LL} \pm J_{\perp}^{RR}\right). \tag{6.1.31}$$

We see that the Hamiltonian only depends on the bosonic field χ_s in certain angles. Performing a rotation of the system we can get rid of χ_s. The rotation we use is [62]: $\mathcal{H}' = UHU^{\dagger}$, $\mathcal{Y}' = U\mathcal{Y}U^{\dagger}$, with $U = \exp(i\chi_s\tau^z)$. Since $\mathcal{Y}$ is proportional to $\nabla\Phi_f(x)$ it is unaffacted by the canonical transformation and we can rewrite the Hamiltonian in the simpler form

$$\begin{aligned}\mathcal{H} &= \frac{\hbar v_F}{4\pi}\sum_{\nu=c,s,f,fs}\int_{-\infty}^{\infty}\left(\nabla\Phi_\nu\right)^2 dx \\ &+\frac{J^+}{\pi a}\tau^y\cos(\chi_{sf}) - \frac{J^-}{\pi a}\tau^x\sin(\chi_{sf}) - \frac{J_{\perp}^{LR}}{\pi a}\tau^x\sin(\chi_f) \\ &+\left(\frac{J_z}{2\pi} - \hbar v_F\right)\nabla\Phi_s(x)\tau^z - h\tau^z.\end{aligned} \tag{6.1.32}$$

The goal of this rotation is not only to make the Hamiltonian look nicer it also makes the equivalent fermionic model quadratic. So we now take a step back to a fermionic description with the use of the refermionisation identities:

$$\psi_\nu(x) = \frac{e^{i\pi d^\dagger d}}{\sqrt{2\pi a}}e^{-i\chi_\nu}, \tag{6.1.33}$$

$$\psi_\nu(x) = \frac{e^{-i\pi d^\dagger d}}{\sqrt{2\pi a}}e^{i\chi_\nu}, \tag{6.1.34}$$

where

$$d = i\tau^x - \tau^y = i\tau^+, \tag{6.1.35}$$

describes the impurity spin. For a spin 1/2 eq. (6.1.35) assures that d and $d^\dagger$ satisfy the anti-commutation relationship $\{d, d^\dagger\} = 1$. The phase factors in eqs. (6.1.33) and (6.1.34) takes care of different species ψ_ν anti-commutation with d and $d^\dagger$, while χ_ν guarantees that

the ψ_ν obey the commutation relations. We now obtain a Hamiltonian in a new fermionic form

$$\begin{aligned}\mathcal{H}' &= i\hbar v_F \sum_{\nu=c,s,f,sf} \int dx \psi_\nu^\dagger(x) \frac{\partial}{\partial x} \psi_\nu(x) dx \\ &+ \frac{J^+}{2\sqrt{2\pi a}} \left(\psi_{sf}^\dagger(0) + \psi_{sf}(0)\right)(d^\dagger - d) \\ &+ \frac{J_\perp^{LR}}{2\sqrt{2\pi a}} \left(\psi_f^\dagger(0) - \psi_f(0)\right)(d^\dagger + d) \\ &+ \frac{J^-}{2\sqrt{2\pi a}} \left(\psi_{sf}^\dagger(0) - \psi_{sf}(0)\right)(d^\dagger + d) \\ &+ \left[h - (J_z - 2\pi v_F) : \psi_s^\dagger \psi_s(0) :\right](d^\dagger d - 1/2),\end{aligned} \tag{6.1.36}$$

and

$$\mathcal{Y}_0' = eV \int \psi_f^\dagger(x) \psi_f(x) dx. \tag{6.1.37}$$

If we set $J_z = 2\pi v_F$ both the non-equilibrium component $\mathcal{Y}'$ and the full Hamiltonian $\mathcal{H}'$ reduce to a quadratic form. In this limit the ψ_c and ψ_s fields de-couple from the d-fermions and we just need to consider ψ_f and ψ_{sf} when we want to study impurity quantities such as the charge and spin current. Restricting our attention to the flavour and spin flavour field we introduce the Fourier transform

$$\psi_\nu^\dagger = \frac{1}{\sqrt{L}} \sum_k \psi_{\nu,k}^\dagger e^{ikx} \qquad \nu = f, sf, \tag{6.1.38}$$

where $k = 2\pi n/L$ and L is the length of the system. The fermionic operators satisfy the standard anti-commutator relationship

$$\{\psi_{\nu,k}^\dagger, \psi_{\nu',k'}\} = \delta_{k,k'} \delta_{\nu,\nu'}. \tag{6.1.39}$$

We now introduce the Majorana Fermions

$$\hat{a} = \frac{d + d^\dagger}{\sqrt{2}}, \qquad \hat{b} = \frac{d^\dagger - d}{i\sqrt{2}}, \tag{6.1.40}$$

which satisfy the relationship $\hat{a}^2 = \hat{b}^2 = 1/2$ instead of zero which is the case for standard fermions. Combining eqs. (6.1.38) and (6.1.40) we can rewrite the Hamiltonian in the form

$$\begin{aligned} \mathcal{H} &= \sum_{\nu=f,sf} \sum_k \epsilon_k \psi^\dagger_{\nu,k} \psi_{\nu,k} - h\hat{a}\hat{b} \\ &+ \frac{J^+}{2\sqrt{2\pi a L}} \sum_k \left(\psi^\dagger_{sf,k} + \psi_{sf,k} \right) \hat{b} \\ &+ \frac{J^{LR}_\perp}{2\sqrt{2\pi a L}} \sum_k \left(\psi^\dagger_{f,k} - \psi_{f,k} \right) \hat{a} \\ &+ \frac{J^-}{2\sqrt{2\pi a L}} \sum_k \left(\psi^\dagger_{sf,k} - \psi_{sf,k} \right) \hat{a}, \end{aligned} \tag{6.1.41}$$

and

$$\mathcal{Y}'_0 = eV \sum_k \psi^\dagger_{f,k} \psi_{f,k}. \tag{6.1.42}$$

6.1.1 Noise in the Toulouse limit

The spin and charge number operators are given by

$$N_c = \frac{1}{2} \sum_\alpha \sum_\sigma \alpha \int dx \psi^\dagger_{\alpha\sigma}(x) \psi_{\alpha\sigma}(x), \tag{6.1.43}$$

$$N_s = \frac{1}{2} \sum_\alpha \sum_\sigma \alpha\sigma \int dx \psi^\dagger_{\alpha\sigma}(x) \psi_{\alpha\sigma}(x), \tag{6.1.44}$$

where $\alpha = L, R = \pm 1$. We now obtain the spin and charge currents from the Heisenberg equation of motion

$$I_{c/s} = -i[\mathcal{H}, N_{c/s}]. \tag{6.1.45}$$

Performing the commutation relation we see that the charge current is given by

$$I_c = \frac{J_\perp^{LR}}{2}\left(\psi_{R\uparrow}^\dagger(t)\psi_{L\downarrow}(t') + \psi_{R\downarrow}^\dagger(t)\psi_{L\uparrow}(t') - \psi_{L\uparrow}^\dagger(t)\psi_{R\downarrow}(t') - \psi_{L\downarrow}^\dagger(t)\psi_{R\uparrow}(t')\right). \quad (6.1.46)$$

The noise is now defined in terms of the charge current

$$S(\omega) = \int_{-\infty}^{\infty} e^{i\omega t}\left[\langle\{I_c(t), I_c(t')\}\rangle - 2\langle I_c\rangle^2\right] dt. \quad (6.1.47)$$

We are now interested in how this original problem maps under the transformation that mapped the original Hamiltonian into the solvable model. Taking a look at the charge current operator it is clear that it maps into

$$i\frac{eJ_\perp^{LR}}{2\hbar\sqrt{\pi aL}}\sum_k\left(\psi_{f,k}^\dagger + \psi_{f,k}\right)\hat{a}, \quad (6.1.48)$$

which can be seen from eq. (6.1.13) and using eqs. (6.1.41). So after mapping the model to a non-interacting model we can now write the current-current correlator in terms of the single particle Green functions $G_{aa}(t,t')$, $G_{fk,a}(t,t')$ and $G_{fk,fk'}(t,t)$. Since much of the analysis of the noise at this stage is similar to previous chapters we will only quote the results and much of the details can be found in [59]. A careful analysis of the current-current correlator gives us

$$\begin{aligned} I^>(t,t') &= \langle I_c(t)I_c(t')\rangle \quad (6.1.49)\\ &= \langle I_c\rangle^2 + \frac{e^2}{\hbar}\Gamma_1\nu_F\frac{1}{L}\sum_{k,k'}\left[G^>_{fk,fk'}(t,t')G^>_{aa}(t,t') - G^>_{fk,a}(t,t')G^>_{a,fk}(t,t')\right]. \end{aligned}$$

Where the single particle Green functions of the different field species are given by

$$G_{fk,fk'}(t,t') = \langle\left(\psi_{f,k}^\dagger(t) + \psi_{f,k}(t)\right)\left(\psi_{f,k'}^\dagger(t') + \psi_{f,k'}(t')\right)\rangle, \quad (6.1.50)$$

and

$$G_{a,fk} = \langle a(t) \left(\psi^{\dagger}_{f,k'}(t') + \psi_{f,k'}(t') \right) \rangle. \tag{6.1.51}$$

We see from eq. (6.1.49) that we get cancellation of the disconnected terms and what remains is to calculate the Green functions explicitly. Since the goal of this chapter is to motivate future work we choose to omit technical details and instead we choose to discuss the results and the physics they imply. For an explicit calculation of the Green functions the interested reader can consult the article written by Schiller and Hershfield [59] and for a discussion of the full counting statistics of this problem see the work by Schmidt [66], Schmidt *et al.* [67] and Gogolin *et al.* [68]

6.1.2 The zero field limit

In the limit where the external magnetic field is zero we have single particle transport processes that involve the magnetic impurity. This implies that the effective charge is bounded from above by the Poisson statistics result $e^* = e$. The noise in this limit is given by [59]

$$\frac{S(0)}{2I_c} = e \left(1 - \frac{\Gamma_1 eV}{\arctan(eV/\Gamma_1)[(eV)^2 + \Gamma_a^2]} \right). \tag{6.1.52}$$

So in the shot noise regime we obtain the Poison statistics result and an effective charge of $e^* = e$

6.1.3 Large field limit

In the large external magnetic field limit we can no longer have single particle spin flip processes of the spin on the impurity since these states would have too high an energy. So the only processes that are allowed are virtual double spin flips that in one step moves two

particles and give us the effective charge $e^* = 2e$. This result is also what is obtained in the shot noise regime in the strong magnetic field limit [59].

6.2 Effective charge

In the previous sections we saw what happens to the effective charge in two extreme limits where only one type of scattering process is responsible for the transport. In this section we will discuss a more realistic situation where we have multiple scattering processes taking place. In a recent article, Sela and co-workers [60] study shot noise in a quantum dot in the Kondo regime. This is done using Nozières Fermi liquid theory [26]. In this work a quantum dot with a symmetric coupling to the leads is considered near the unitary limit and the mixture of left and right movers is controlled by the applied bias voltage. Due to the left and right symmetry we can write the low energy Hamiltonian of the system[19, 69]

$$\begin{aligned}\mathcal{H} &= \sum_{k\sigma} \xi_k \psi^\dagger_{k\sigma}\psi_{k\sigma} - \frac{\alpha}{2\pi\nu T_K}\sum_{k,k'\sigma}(\xi_k + \xi_{k'})\psi^\dagger_{k\sigma}\psi_{k'\sigma} \\ &\quad + \frac{\beta}{\pi\nu T_K}\sum_{k_1,k_2,k_3,k_4} \psi^\dagger_{k_1\uparrow}\psi_{k_2\uparrow}\psi^\dagger_{k_3\downarrow}\psi_{k_4\downarrow},\end{aligned} \tag{6.2.1}$$

in terms of left and right moving electrons $\psi_{k\sigma} = \frac{1}{\sqrt{2}}(L_{k\sigma} + R_{k\sigma})$. The term that is proportional to α is proportional to the energy of the phase shift and the terms proportional to β describes the quasiparticle interactions. The current transmitted through the dot consists of two parts: the maximal unitary limit $I_u = 2\frac{e^2}{h}V$ and the back scattering term I_b. The back scattering terms are due to the interaction of the dot. In [60] the effective charge is calculated in the back scattering current through the relation

$$e^* = \frac{S(0)}{2I_b}, \tag{6.2.2}$$

where S(0) is the shot noise. Performing this longwinded calculation one obtains the following result for the current and the noise [60]:

$$I = I_u - I_b = \frac{2e^2}{h} V \left[1 - \frac{\alpha^2 + \beta^2}{12} \left(\frac{V}{T_K} \right)^2 \right], \tag{6.2.3}$$

$$S(0) = \frac{4e^3}{h} V \frac{\alpha^2 + \beta^2}{12} \left(\frac{V}{T_K} \right)^2. \tag{6.2.4}$$

Inserting eqs. (6.2.3) and (6.2.4) into eq. (6.2.2) we obtain

$$e^* = \frac{\alpha^2 + 9\beta^2}{\alpha^2 + 5\beta^2}. \tag{6.2.5}$$

It is a central result in Nozières Fermi Liquid theory that for the Kondo effect $\alpha = \beta$ and for this value we obtain the Wilson ratio

$$W = \left(\frac{\delta\chi}{\chi} \right) / \left(\frac{\delta C_v}{C_v} \right) = 1 + \frac{\beta}{\alpha} = 2, \tag{6.2.6}$$

where χ is the susceptibility and C_v is the specific heat. The Wilson ratio is very useful for characterisation of strongly correlated Fermi liquids, for a detailed discussion of this topic see [70]. These values for α and β give us the effective charge $e^* = 5/3$. This result can be understood by looking at the part of the Hamiltonian related to β:

$$H_\beta = \frac{\beta}{\pi\nu T_K} \sum_{k_1,k_2,k_3,k_4} \psi^\dagger_{k_1} \psi_{k_2} \psi^\dagger_{k_3} \psi_{k_4}. \tag{6.2.7}$$

Splitting the field ψ into left and right movers,

$$\psi_{k\sigma} = \frac{1}{\sqrt{2}} \left(\psi_{Lk\sigma} + \psi_{Rk\sigma} \right), \tag{6.2.8}$$

we obtain a Hamiltonian that contains scattering processes that will back scatter 0, 1 and 2 particles. We choose the term

$$\sum \psi^{\dagger}_{Lk_1\uparrow}\psi_{Rk_2\uparrow}\psi^{\dagger}_{Lk_3\downarrow}\psi_{Rk_4\downarrow}, \tag{6.2.9}$$

that backscatters two right movers into two left movers. The contribution this process will make to the backscattering current is $I_{2\beta} = 2e\Gamma_{2\beta}$, where we have introduced the scattering rate

$$\Gamma_{2\beta} = \frac{2\pi}{h}\sum_{k_1k_2k_3k_4} |\langle\psi_{Lk_1\uparrow}\psi^{\dagger}_{Rk_2\uparrow}\psi_{Lk_3\downarrow}\psi^{\dagger}_{Rk_4\downarrow}H_{\beta}\rangle|^2\delta(\xi_{k_1}+\xi_{k_2}-\xi_{k_3}-\xi_{k_4}). \tag{6.2.10}$$

Now using the relationships

$$\langle\psi_{Lk\sigma}\psi^{\dagger}_{Lk'\sigma'}\rangle = \delta_{kk'}\delta_{\sigma\sigma'}(1-f_L(\xi_k)), \tag{6.2.11}$$

$$\langle\psi_{Rk\sigma}\psi^{\dagger}_{Rk'\sigma'}\rangle = \delta_{kk'}\delta_{\sigma\sigma'}(f_R(\xi_k)), \tag{6.2.12}$$

we obtain the following two particle backscattering contribution to the backscattering current

$$I_{2\beta} = \frac{e^2}{h}\frac{2}{3}\left(\frac{V}{T_K}\right)^2 V\beta^2. \tag{6.2.13}$$

In a similar fashion we can now obtain the contribution from the single particle scattering processes that consist of the elastic processes from the terms proportional to α,

$$I_{1\alpha} = \frac{e^2}{h}\frac{1}{6}\left(\frac{V}{T_K}\right)^2 V\alpha^2, \tag{6.2.14}$$

in the Hamiltonian and the inelastic processes proportional to β,

$$I_{1\beta} = \frac{e^2}{h}\frac{1}{6}\left(\frac{V}{T_K}\right)^2 V\beta^2. \tag{6.2.15}$$

Since the rates are very low, $(V/T_K)^2 \ll 1$, we can assume that the rates are uncorrelated and we get the total contribution to the noise given by

$$S(0) = 2e(I_{1\alpha} + I_{1\beta} + 2I_{2\beta}), \tag{6.2.16}$$

which gives an effective charge

$$e^* = e\frac{\frac{\alpha^2}{6} + \frac{\beta^2}{6} + 2\frac{2\beta^2}{3}}{\frac{\alpha^2}{6} + \frac{\beta^2}{6} + \frac{2\beta^2}{3}}. \tag{6.2.17}$$

This result give us $e^* = 5e/3$ for $\alpha = \beta$. Sela and co-workers [60] claim that this is a universal result and more general then the Wilson ratio. An important question is of course if there exists a relationship between the Wilson ratio and the effective charge. In a recent work Fujii [61] finds the following relationship between the shot noise and the Wilson ratio:

$$e^* = e\left(1 + \frac{4(W-1)^2}{1 + 5(W-1)^2}\right). \tag{6.2.18}$$

for a quantum dot. For our system with the Wilson ratio 2 thus give the expected $e^* = 5e/3$.

6.3 Magnetic impurities in a Luttinger liquid

In the previous two chapters we have introduced the theory necessary to calculate the effective charge in 1D systems and discussed the current state of research in this field. We will now close this chapter by suggesting a new direction of research for this field. We have earlier in this chapter studied a model with non-interacting one-dimensional electrons coupled to a impurity. The next step will be to add interactions to the system and study a Luttinger liquid connected to a magnetic impurity [71, 72, 73]. This model was originally studied by Furusaki and Nagaosa [72] and Fröjd and Johannesson [73]. The Luttinger liquid

is described by the Hamiltonian

$$\mathcal{H}_{LL} = \frac{\nu_\rho}{4\pi}\int dx\left[\frac{1}{K_c}\left(\frac{d\Phi_c}{dx}\right)^2 + K_c\left(\frac{d\Phi_f}{dx}\right)^2\right] \tag{6.3.1}$$

$$+\frac{\nu_\sigma}{4\pi}\int dx\left[\frac{1}{K_s}\left(\frac{d\Phi_s}{dx}\right)^2 + K_s\left(\frac{d\Phi_{sf}}{dx}\right)^2\right], \tag{6.3.2}$$

where

$$\Phi_c = \frac{1}{2}\left(\Phi_{L\uparrow} + \Phi_{L\downarrow} + \Phi_{R\uparrow} + \Phi_{R\downarrow}\right), \tag{6.3.3}$$

$$\Phi_s = \frac{1}{2}\left(\Phi_{L\uparrow} - \Phi_{L\downarrow} + \Phi_{R\uparrow} - \Phi_{R\downarrow}\right), \tag{6.3.4}$$

$$\Phi_{cf} = \frac{1}{2}\left(\Phi_{L\uparrow} + \Phi_{L\downarrow} - \Phi_{R\uparrow} - \Phi_{R\downarrow}\right), \tag{6.3.5}$$

$$\Phi_{sf} = \frac{1}{2}\left(\Phi_{L\uparrow} - \Phi_{L\downarrow} - \Phi_{R\uparrow} + \Phi_{R\downarrow}\right), \tag{6.3.6}$$

are bosonic fields and K_c and K_s are the Luttinger liquid parameters that control the charge and spin sectors respectively. We now attach an impurity spin (S=1/2) at the origin. We now need to consider two types of Kondo exchange couplings: forward and backward scattering. These two processes are described by the Hamiltonian

$$\mathcal{H}_{imp} = \frac{J_F}{2}\vec{S}\cdot\left[\psi^\dagger_{R\sigma}(0)\vec{\sigma}_{\alpha\beta}\psi_{R\beta}(0) + \psi^\dagger_{L\sigma}(0)\vec{\sigma}_{\alpha\beta}\psi_{L\beta}(0)\right]$$

$$+\frac{J_B}{2}\vec{S}\cdot\left[\psi^\dagger_{R\sigma}(0)\vec{\sigma}_{\alpha\beta}\psi_{L\beta}(0) + \psi^\dagger_{L\sigma}(0)\vec{\sigma}_{\alpha\beta}\psi_{R\beta}(0)\right], \tag{6.3.7}$$

where $\vec{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$ are the Pauli matrices. In this model the Wilson ratio has been calculated by Fröjdh and Johannesson [73] to be

$$W = \frac{4}{3}\left(1 + \frac{\nu_c}{\nu_s}\right), \tag{6.3.8}$$

where ν_c is the velocity of charge excitations and ν_s the spin velocity. This model is also solvable in the Toulouse limit with the use of bosonisation and will provide a good test of

how general the results of Sela et al. [60] really are. Therefore we believe that it is a natural extension of the theory presented in this thesis and provides a good project for future work.

6.4 Conclusion

In this chapter we have studied how to deal with Kondo impurities in a one dimensional system in the Toulouse limit. This was done by bosonisation and Emery-Kivelson rotation. Then the system was refermionised and we obtained a solvable model in the Toulouse limit. We then discussed the limits of this model when we obtain an effective charge of e and 2e and what kind of processes that are responsable for this. Finally we discussed the recent result where the effective charge of, $e^* = 5e/3$, of a Kondo dot was derived, and we suggested a system where a further investigation of this result would be interesting.

Chapter 7

CONCLUSION

In this thesis the main topic has been Coulomb blockaded quantum dots and the goal has been to calculate the noise in this system. The motivation for this work is that most work in this systems has been done using the single particle tunneling "orthodox" theory. This theory is based on a classical master equation approach. The validity of this approach at the peak of conductance is unclear. The reason for this is that we have two strongly interacting charging levels and to figure out if the results from the "orthodox" theory are valid, a full quantum treatment using the Keldysh technique is necessary.

The first part of this thesis has been dedicated to understanding the basic theory of Coulomb blockaded quantum dots and the methods that today are accessible to study the properties of these systems. Much of this work is a direct extension to the work done by Sedlmayr *et al.* [1] where the TDoS is calculated for a Coulomb blockaded quantum dot. This method developed and used in this work runs into problems when we start to study noise. The reason for this is that the bosonic field introduced in the Hubbard-Stratonovich transformation makes it extremely difficult to calculate the two particle Green functions of the dot that appear in the noise. We therefore devote chapter 3 to introduce the method developed by Sedlmayr *et al.* [1] and use it to calculate the TDoS. We then go on and develop a new method to calculate the single particle Green functions and the TDoS without the need to

perform a Hubbard-Stratonovich transformation. The key in this new method is to rewrite the Hamiltonian such that we can separate parts depending on the particle number of the dot and the distribution of the levels of the dot. We then attach the tunneling to one of the levels of the dot and we can treat the problem as a resonant level problem and the interactions play the role of weights to the charge states. This is easily generalised to two particle Green functions and we can now treat most of the problem of the noise in the Coulomb blockaded quantum dot as a resonant level and the interactions once again enter as weights to the charge states.

Using this new method we can rewrite the noise calculations in two parts: one that treats the interactions of the charge states and another that turns in to a resonant level model. For the second problem we can use the results of section 2.5 with small modifications. The main result is that we succeed in finding an analytic expression for the noise valid in the region $\Gamma \ll \Delta \ll T \ll E_c$. We have checked the result in a couple of trivial limits such as zero bias and zero temperature and we have found complete agreement with the known results in these limits. In the shot noise regime $T \ll eV$ at the peak of conductance we obtain the same Fano factor, $f = (\Gamma_1^2 + \Gamma_2^2)/\Gamma^2$ as the master equation approach. We also obtain results that are valid in the intermediate regime from the linear response regime up to the shot noise noise regime.

The final chapter was motivated by recent work by Sela *et al.*[60] where it is shown that the effective charge in a Kondo dot is $e^* = 5/3$. There have also been generalisations of this work made by Fujii [61] where the effective charge as a function of the Wilson ratio is calculated. This is claimed to be a universal result and we introduce a new system where we think it would be interesting to study if this is universal or not in a future project. We believe that a natural choice for a generalisation of the results by Sela *et al.* [60] is a Luttinger liquid with a magnetic impurity. The reason this is a good choice is that it is a strongly interacting

system in 1D where we can find an exact solution of the problem.

Appendix A

CURRENT IN A QUANTUM DOT

In this appendix we will derive the general current expression in a quantum dot [38, 74, 75]. This expressions will be used in chapter 4 as a reference point when we calculate the noise in chapter 4. We start from the current operator

$$I_L = -ie\langle[\mathcal{H}, N_L]\rangle, \tag{A.0.1}$$

where

$$N_L = \sum_k c_{kL}^\dagger c_{kL}. \tag{A.0.2}$$

The Hamiltonian $\mathcal{H}$ consists of three parts: the standard non-interacting leads H_0, the tunneling Hamiltonian, H_T, and the central region with a non-interacting kinetic part and the interactions of the central region, $\mathcal{H}_{cen}$. Since $\mathcal{H}_0$ and $\mathcal{H}_{cen}$ commute with N_L the current obtains the form

$$I_L = ie\sum_k \left[t_{k\alpha}\langle c_{k\alpha}^\dagger d_n\rangle - t_{k\alpha}^*\langle d_n^\dagger c_{k\alpha}\rangle\right]. \tag{A.0.3}$$

Performing a S-matrix expansion in the same way as in section 2.5 but to first order in $\mathcal{H}_T$, we obtain the following expression for the current in terms of single particle Green functions

of the non-interacting leads and the interacting central region,

$$G_{nk\alpha}(t,t') = \sum_m \int G_{nm}(t,t_1) t^*_{k\alpha m} g_{k\alpha}(t_1,t'). \tag{A.0.4}$$

Using the analytic continuation rules we obtain

$$G^<_{nk}(t,t') = \sum_m \int dt_1 \left[G^R_{nm}(t,t_1) G^<_{k\alpha}(t_1,t') + G^<_{nm}(t,t_1) G^A_{k\alpha}(t_1,t') \right]. \tag{A.0.5}$$

Performing the Fourier transform we obtain the result in frequency space

$$G^<_{nk}(\varepsilon) = \sum_m G^R_{nm}(\varepsilon) G^<_{k\alpha}(\varepsilon) + G^<_{nm}(\varepsilon) G^A_{k\alpha}(\varepsilon). \tag{A.0.6}$$

Inserting the non-interacting Green functions of the leads and using the relationship between the single particle Green functions, we find

$$I_L = ie \int \frac{d\varepsilon}{2\pi} \mathrm{Tr} \left(\Gamma_L \left\{ \mathbf{G}^<(\varepsilon) + f_L(\varepsilon) [\mathbf{G}^R(\varepsilon) - \mathbf{G}^A(\varepsilon)] \right\} \right). \tag{A.0.7}$$

Using $G^K = G^> + G^<$ we can write this also in the equivalent form

$$I_\alpha = e\Gamma_\alpha \int_{-\infty}^{\infty} \frac{d\varepsilon}{4\pi} \left(\mathrm{Tr} \left\{ \mathbf{G}^K(\varepsilon) - [1 - 2f_\alpha(\varepsilon)] \left[\mathbf{G}^R(\varepsilon) - \mathbf{G}^A(\varepsilon) \right] \right\} \right). \tag{A.0.8}$$

In the steady state the current will be uniform and we can write the current in the form $I = (I_L + I_L)/2 = (I_L - I_R)/2$. Using this relationship we can write the current in the standard expression for the dc current

$$I = ie \int \frac{d\varepsilon}{2\pi} \mathrm{Tr} \left\{ (\Gamma_L - \Gamma_R) \mathbf{G}^<(\varepsilon) + (\Gamma_L f_L(\varepsilon) - \Gamma_R f_R(\varepsilon)) [\mathbf{G}^R(\varepsilon) - \mathbf{G}^A(\varepsilon)] \right\}. \tag{A.0.9}$$

Using current conservation, $I_L = -I_R$, we can rewrite this expression on the form

$$I = ie \int \frac{d\varepsilon}{2\pi} \left[f_L(\varepsilon) - f_R(\varepsilon) \right] \mathcal{T}(\varepsilon), \tag{A.0.10}$$

where

$$\mathcal{T}(\varepsilon) = \mathrm{Tr} \left\{ \frac{\Gamma_1 \Gamma_2}{\Gamma_1 + \Gamma_2} \left(\mathbf{G}^R(\varepsilon) - \mathbf{G}^A(\varepsilon) \right) \right\}. \tag{A.0.11}$$

Appendix B

FULL COUNTING STATISTICS

In this thesis we have focused on studying the noise and average current in strongly interacting systems. These are both properties that can be extracted from the full counting statistics of the system [76, 77]. So in this appendix we will discuss some basic properties of full counting statistics such that it will be easier to understand and compare the results in this thesis to the literature. This is especially important in chapter 6 where the effective charge is studied, as this is a property that can be calculated both from the noise/second cumulant and as the square root of the third cumulant [78].

The starting point is to introduce the characteristic function of a simple electron system where the transmission is in the region $0 < T_n < 1$. We introduce this without a proof since the derivation is rather involved [76] and we believe the derivation itself will not add much to the understanding of what information can be extracted from the first few cumulants. For this system the characteristic function, also known as the Levitov-Lesovik formula, is given by

$$\begin{aligned} \ln \Lambda(\chi) &= 2\Delta t \int \frac{d\varepsilon}{2\pi\hbar} \sum_n \ln \left\{ 1 + T_n \left(e^{i\chi} - 1 \right) f_L(\varepsilon) \left(1 - f_R(\varepsilon) \right) \right. && \text{(B.0.1)} \\ &\quad \left. + T_n \left(e^{i\chi} - 1 \right) f_R(\varepsilon) \left(1 - f_R(\varepsilon) \right) \right\}. && \text{(B.0.2)} \end{aligned}$$

The logarithm in eq. (B.0.2) comes with certain assumptions since it implies sum over the channels which means that transmissions in different channels are independent, and the integral over the energy implies that electrons are transmitted independently in different energy intervals. The first derivative gives us the first cumulant as follows:

$$\langle q \rangle = e \left. \frac{\partial \ln \Lambda}{\partial (i\chi)} \right|_{\chi=0} = \frac{2e\Delta t}{2\pi\hbar} \sum_n \int d\varepsilon T_n(\varepsilon) \left(f_L(\varepsilon) - f_R(\varepsilon) \right). \tag{B.0.3}$$

A quick comparison with the Landauer formula [6, 38], gives the relationship

$$\langle q \rangle = \langle I \rangle \Delta t. \tag{B.0.4}$$

The second cumulant is given by

$$\begin{aligned} \langle\langle q^2 \rangle\rangle = \frac{e^2 \Delta t}{\pi\hbar} \sum_n \int d\varepsilon \, \{ & T_n(\varepsilon) \left[f_L(\varepsilon)(1 - f_L(\varepsilon)) + f_R(\varepsilon)(1 - f_R(\varepsilon)) \right] \\ & + T_n(\varepsilon)(1 - T_n(\varepsilon))(f_L(\varepsilon) - f_R(\varepsilon))^2 \big\} . \end{aligned} \tag{B.0.5}$$

To understand what this means we start by investigating the equilibrium situation, i.e. $f_L = f_R$. The second cumulant now has the form

$$\langle\langle q^2 \rangle\rangle = \frac{2_s e^2 k_B T}{\pi\hbar} \sum_n T_n. \tag{B.0.6}$$

Comparing this to eq. (1.4.5) we obtain the relationship

$$\langle\langle q^2 \rangle\rangle = \frac{\Delta t S(0)}{2}. \tag{B.0.7}$$

In the shot noise limit, $k_B T \ll eV$, we obtain

$$\langle\langle q^2 \rangle\rangle = \Delta t G e V \sum_n T_n (1 - T_n). \tag{B.0.8}$$

If we compare this with the standard result for shot noise [7], we obtain the relationship

$$\langle\langle q^2\rangle\rangle = \frac{\Delta t S(0)}{2}. \tag{B.0.9}$$

The third cumulant in the shot noise regime is given by

$$\langle\langle q^3\rangle\rangle = e^2 V G \Delta t \sum_n T_n(1 - T_n). \tag{B.0.10}$$

If we now take the low transmission limit, $T \ll 1$, this can be reduced to the simple relationship

$$\langle\langle q^3\rangle\rangle = e^2 \Delta t \langle I\rangle. \tag{B.0.11}$$

It has been suggested by Levitov and Reznikov [78] that the third cumulant is better to use than the shot noise when it comes to detecting the charge of quasi-particles, through the relation

$$(e^*)^2 = \frac{\langle\langle q^3\rangle\rangle}{\langle q\rangle}. \tag{B.0.12}$$

The reason that this is a better measure now that the third cumulant is experimentally accessible [78, 79, 80] is that it is less sensitive to thermal fluctuations.

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